Solving Heterogeneous Agent Model with Projection and Perturbation

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Solving Incomplete Market Models with Hetero Agents

- Projection
    - DR2010-JEDC (Exact Aggregation/Xpa) *
    - AAD2008-JEDC
    - AAD2010-JEDC
    - Reiter2010-JEDC
- Perturbation
    - KKK2010-JEDC
    - PR2006-WP
- Hybrid:
    - Projection and Simulation (i.e., Krusell-Smith Algorithm)
        - KS1998-JPE
        - MMV2010-JEDC (KS- Stochastic Simulation). *
        - Young2010-JEDC (KS- Non-Stochastic Simulation 2)
    - Projection and Perturbation
        - Reiter2009-JEDC*
        - Winberry2018-QE*
- Continuous-time: AKMWW2018-NBER Macro Annual

Theoretical Model: Recursive competitive equilibrium

Households:

$$c(\varepsilon, k; m, a)^{-\sigma} = h(\varepsilon, k; m, a) + \beta E[c(\varepsilon', k'; m', a')^{-\sigma}(1 - \delta + r')] \quad (1)$$

$$c(\varepsilon, k; m, a) = [(1-\tau)\varepsilon_t + \mu(1-\varepsilon_t)]w(m, a) + [r(m, a) + 1 - \delta]k - k'(\varepsilon, k; m, a) \quad (2)$$

Firms:

$$w = (1 - \alpha)a_t(\frac{K_t}{L_t})^{\alpha} \quad (3)$$

$$r = \alpha a_t(\frac{K_t}{L_t})^{\alpha - 1} \quad (4)$$

Government:

$$\tau_t = \frac{\mu u_t}{L_t} = \frac{\mu(1 - L_t)}{L_t}$$

Theoretical Model

- Aggregate states:

$$log(a_{t+1}) = \rho_a log(a_t) + \sigma_a \omega_{t+1}, \omega \sim N(0,1) \quad (5)$$

- Idiosyncratic states: employed / unemployed:
  - $\varepsilon_t = 1$, if employed;
  - $\varepsilon_t = 0$, if unemployed;
- Transition probabilities (constant over time[1])

e/e'	u	e
u	$\pi(u\|u)$	$\pi(u\|e)$
e	$\pi(e\|u)$	$\pi(e\|e)$

- Evolution of distribution: for all measurable sets Δ_k

$$m'(m,a) = \sum_{\tilde{\varepsilon}} \pi(\varepsilon|\tilde{\varepsilon}) \int 1\{k'(\tilde{\varepsilon}, k; m, a) \in \Delta_k\} m(\tilde{\varepsilon}, dk) \quad (6)$$

[1]This implies L_t is constant over time, so is tax rate τ_t.

Computational Challenges

"Infinite-dimensional Fixed Point Problem":

$$c(\varepsilon, k; m, a)^{-\sigma} = h(\varepsilon, k; m, a) + \beta E[c(\varepsilon', k'; m', a')^{-\sigma}(1 - \delta + r')]$$

where optimal consumption $c(\varepsilon, k; m, a) =$

$$kr(m, a) + [(1 - \tau)\varepsilon_t + \mu(1 - \varepsilon_t)]w(m, a) + (1 - \delta)k - k'(\varepsilon, k; m, a)$$

and m is the (joint) distribution of capital and employment status (usually an infinite-dimensional object).

Handle the Challenges

Individual problem:

- idiosyncratic shocks (uncertainty): large (0 or 1);
- nonlinear and high dimensional individual problem
- perturbation (local solution) probably a bad idea
- projection is needed (as in KS algorithm etc.)

Aggregate problem:

- aggregate shocks (uncertainty): relatively small (1 s.d.);
- linear or almost linear aggregate problem
- perturbation (local solution) probably works

Solution: Projection+Perturbation.

Projection + Perturbation Algorithm

- Step 1: Approximate the model's equilibrium objects - the distribution, law of motion, factor prices, decision rules (often infinite-dimensional)- using *finite dimensional global approximations* w.r.t. individual state variables.
- Step 2: Compute the stationary equilibrium of the approximated model without aggregate shocks but still with idiosyncratic shocks.
- Step 3: Compute the aggregate dynamic of the approximated model by perturbing it around the stationary equilibrium.

Step 1: Approximation

a. Approximate distribution of household over capital holding (k) with $g_{\varepsilon,t}(k) \simeq$

$$g^0_{\varepsilon,t} exp\{g^1_{\varepsilon,t}[k-m^1_{\varepsilon,t}]+g^2_{\varepsilon,t}[(k-m^1_{\varepsilon,t})^2-m^2_{\varepsilon,t}]+...+g^J_{\varepsilon,t}[(k-m^1_{\varepsilon,t})^J-m^J_{\varepsilon,t}]\} \quad (7)$$

where J is the order of approximation, $g^j_{\varepsilon,t}$ are parameters, $m^j_{\varepsilon,t}$ are centralized moments of distribution.[2]

$$m^1_{\varepsilon,t} = \int k g_{\varepsilon,t}(k) dk \quad (8)$$

$$m^j_{\varepsilon,t} = \int (k - m^1_{\varepsilon,t})^j g_{\varepsilon,t}(k) dk, \text{for j=2,3,...,J.} \quad (9)$$

Issue: occasionally binding constraint

[2] In practice we approximate the integrals using Gauss-Legendre quadrature.

Step 1: Approximation

a. (cont-) Approximate distribution of household over capital holding (k) *at* the borrowing constraint ($k=\underline{k}$).
→ The distribution of household features a *positive* mass at $\underline{k}$
→ We denote the mass at constraint with productivity ε as $\hat{m}_\varepsilon$.
Law of motion of the mass

$$\hat{m}_{\varepsilon,t+1} = \frac{1}{\pi(\varepsilon)}[\sum_{\tilde{\varepsilon}} \hat{m}_{\tilde{\varepsilon},t}\pi(\tilde{\varepsilon})\pi(\varepsilon|\tilde{\varepsilon})1\{k'(\tilde{\varepsilon},\underline{k}) = \underline{k}\}+$$

$$\sum_{\tilde{\varepsilon}}(1-\hat{m}_{\tilde{\varepsilon},t})\pi(\tilde{\varepsilon})\pi(\varepsilon|\tilde{\varepsilon})\int 1\{k'(\tilde{\varepsilon},k) = \underline{k}\}g_{\tilde{\varepsilon},t}(k)dk] \tag{10}$$

where $\pi(\varepsilon)$ is the mass of households with prod. ε;
$g_{\varepsilon,t}(k)$ is the p.d.f of households with $k > \underline{k}$ as defined before.

Step 1: Approximation

b. Approximate law of motion of *distribution* by law of motion of *moments*.[3]

$$m^1_{\varepsilon,t+1} = \frac{1}{\pi(\varepsilon)}[\sum_{\tilde{\varepsilon}} \hat{m}_{\tilde{\varepsilon},t}\pi(\tilde{\varepsilon})\pi(\varepsilon|\tilde{\varepsilon})k'(\tilde{\varepsilon},\underline{k})+$$

$$\sum_{\tilde{\varepsilon}}(1-\hat{m}_{\tilde{\varepsilon},t})\pi(\tilde{\varepsilon})\pi(\varepsilon|\tilde{\varepsilon})\int k'(\tilde{\varepsilon},k)g_{\tilde{\varepsilon},t}(k)dk] \tag{11}$$

$$m^j_{\varepsilon,t+1} = \frac{1}{\pi(\varepsilon)}[\sum_{\tilde{\varepsilon}} \hat{m}_{\tilde{\varepsilon},t}\pi(\tilde{\varepsilon})\pi(\varepsilon|\tilde{\varepsilon})[k'(\tilde{\varepsilon},\underline{k})-m^1_{\varepsilon,t+1}]^j+$$

$$\sum_{\tilde{\varepsilon}}(1-\hat{m}_{\tilde{\varepsilon},t})\pi(\tilde{\varepsilon})\pi(\varepsilon|\tilde{\varepsilon})\int [k'(\tilde{\varepsilon},k)-m^1_{\varepsilon,t+1}]^j g_{\tilde{\varepsilon},t}(k)dk] \tag{12}$$

[3] In practice we approximate the integrals using Gauss-Legendre quadrature.

Step 1: Approximation

c. Compute aggregate capital stock from *approximated* distribution:

$$K_t = \sum_{\varepsilon} \pi(\varepsilon) \sum_{j=1}^{J} \omega_j k_j g_{\varepsilon,t}(k_j)$$

And factor prices:

$$w = (1-\alpha)a_t(\frac{K_t}{L_t})^{\alpha} \quad (13)$$

$$r = \alpha a_t(\frac{K_t}{L_t})^{\alpha-1} \quad (14)$$

Step 1: Approximation

d. Approximate decision rules:
Approximate Conditional Expectation with Chebyshev Polynomials:

$$E[\beta(1-\delta+r')c(\varepsilon',k';m',a')^{-\sigma}] \simeq exp\{\sum_{i=1}^{I}\theta_{\varepsilon i,t}T_i\xi(k)\}$$

Optimal equation:

$$c(\varepsilon,k;m,a)^{-\sigma} = h(\varepsilon,k;m,a) + exp\{\sum_{i=1}^{I}\theta_{\varepsilon i,t}T_i\xi(k)\} \quad (15)$$

To get the parameters, we approximate households' optimality condition using collocation:

$$exp\{\sum_{i=1}^{I}\theta_{\varepsilon i,t}T_i\xi(k)\} = E[\beta(1-\delta+r')\hat{c}(\varepsilon',\hat{k}';m',a')^{-\sigma}] \quad (16)$$

Step 1: Approximate Model

Approximate model is characterized by:

- Aggregate Law of motion (a_t): Equation (5)
- Distribution LM: [Equation (6) $\rightarrow$] Equation (11) and (12)
- Decision rule: [Equation (1) $\rightarrow$] Equation (15) or (16)
- Budget constraint: Equation (2)
- Factor prices: [Equation (3) and (4) $\rightarrow$] Equation (13) and (14)
- Approximate Moment: Equation (8) and (9)

Define a residual function based on the approximate model:

$$E_t[f(y_t, y_{t+1}, x_t, x_{t+1}; \chi)] = 0 \tag{17}$$

Step 2: Stationary Equilibrium

The stationary equilibrium is a system of (nonlinear) equations:

$$f(y^*, y^*, x^*, x^*; 0) = 0 \tag{18}$$

(This can be very difficult to solve due to the large size!)
Author's strategy: write s.s. in terms of K.

- Compute factor prices: **r** and **w**. (by assumption L is constant)
- Solve the approximated expectation term (θ)
- Solve for moments **m** and parameters **g**.
- update aggregate capital K' from equation:

$$K' = \sum_{\varepsilon} \pi(\varepsilon) \sum_{j=1}^{J} \omega_j k_j' g_{\varepsilon}(k_j) \tag{19}$$

- return K'-K and solve for a zero of this equation

Step 3: Perturbation

- Import the model to **Dynare** [4], and **Dynare** will:
- differentiate these equations;
- evaluate them at steady state;
- solve the resulting system at first order;
- perform default analysis.

[4] This step is not trivial. Dynare does not accept matrix expressions used heavily in the matlab codes to solve steady state. We need to re-write teh matrix expressions as loops over scalar variables using Dynare's macro-processor.
For details on macro-processor, see: http://www.dynare.org/manual/index$_3$7*.html*; *or* at: http://www.dynare.org/DynareShanghai2013/macroprocessor.pdf

Reference and Further Reading

Reference

- Winberry, T. (Forthcoming). A toolbox for solving and estimating heterogeneous agent macro models. Quantitative Economics.

which partly builds upon the algorithm in:

- Reiter, M. (2009). Solving heterogeneous-agent models by projection and perturbation. Journal of Economic Dynamics and Control, 33(3), 649-665.
- Algan, Y., Allais, O., Den Haan, W. J. (2010). Solving the incomplete markets model with aggregate uncertainty using parameterized cross-sectional distributions. Journal of Economic Dynamics and Control, 34(1), 59-68.

Reference and Further Reading

Further Reading for P+P Algorithm/Application

- (review) Terry, S. J. (2017). Alternative methods for solving heterogeneous firm models. Journal of Money, Credit and Banking, 49(6), 1081-1111.
- (review) Algan, Y., Allais, O., Den Haan, W. J., Rendahl, P. (2014). Solving and simulating models with heterogeneous agents and aggregate uncertainty. In Handbook of Computational Economics (Vol. 3, pp. 277-324). Elsevier.
- (application) Khan, A., Thomas, J. K. (2008). Idiosyncratic shocks and the role of nonconvexities in plant and aggregate investment dynamics. Econometrica, 76(2), 395-436.