Week 4: Solving Hetero-Agent Model with KS Algorithm

Computation Study Group

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Solving Incomplete Market Models with Hetero Agents

- Projection
  - DR2010-JEDC (Exact Aggregation/Xpa) *
  - AAD2008-JEDC
  - AAD2010-JEDC
  - Reiter2010-JEDC
- Perturbation
  - KKK2010-JEDC
  - PR2006-WP
- Hybrid:
  - Projection and Simulation (i.e., Krusell-Smith Algorithm)
    - KS1998-JPE
    - MMV2010-JEDC (KS- Stochastic Simulation). *
    - Young2010-JEDC (KS- Non-Stochastic Simulation 2)
  - Projection and Perturbation
    - Reiter2009-JEDC*
    - Winberry2018-QE*
- Continuous-time: AKMWW2018-NBER Macro Annual

Environment: DJJ2010-JEDC

$$c_i^{-\gamma} = h_i + \beta E[(c_i')^{-\gamma}(1 - \delta + r')] \tag{1}$$

$$c_i + k_i' = k_i r + [(1 - \tau_t)\varepsilon_t + \mu(1 - \varepsilon_t)]w + (1 - \delta)k_i \tag{2}$$

$$k' \geq 0 \tag{3}$$

$$hk' = 0 \tag{4}$$

$$w = (1 - \alpha)a_t(\frac{K_t}{L_t})^{\alpha} \tag{5}$$

$$r = \alpha a_t(\frac{K_t}{L_t})^{\alpha - 1} \tag{6}$$

$$\tau_t = \frac{\mu u_t}{L_t} = \frac{\mu(1 - L_t)}{L_t} \tag{7}$$

Environment: DJJ2010-JEDC

- Transition probabilities: (Table 2)

$s, e/s'e'$	b, u	b, e	g, u	g, e
b, u	0.525	0.35	0.03125	0.09375
b, e	0.038889	0.836111	0.002083	0.122917
g, u	0.09375	0.03125	0.291667	0.583333
g, e	0.009115	0.115885	0.024306	0.850694

- Aggregate states: bad / good:
  - $a_t = 1 + \Delta$, if good;
  - $a_t = 1 - \Delta$, if bad.
- Idiosyncratic states: employed / unemployed:
  - $\varepsilon_t = 1$, if employed;
  - $\varepsilon_t = 0$, if unemployed;

Computational Challenges

Euler Equation (policy function):

$$c(\varepsilon, k; m, a)^{-\sigma} = h(\varepsilon, k; m, a) + \beta E[c(\varepsilon', k'; m', a')^{-\sigma}(1 - \delta + r')]$$

where optimal consumption $c(\varepsilon, k; m, a) =$

$r(m, a)k + [(1 - \tau_t(m, a))\varepsilon_t + \mu(1 - \varepsilon_t)]w(m, a) + (1 - \delta)k - k'(\varepsilon, k; m, a)$

and m is the (joint) distribution of capital and employment status (usually an infinite-dimensional object).

Computational Challenges

- decisions of each heterogeneous agent depend on r and w.
- r and w depend on the aggregate capital stock;
- aggregate capital stock is determined by cross-sectional capital holding of all heterogeneous agents;
- capital distribution is a state variable, and
- capital distribution is typically an infinite-dimensional object
- complicated fixed point problem: each agent's saving decision depends on his expectation on the dynamics of distribution; the dynamics of distribution depend on agent's saving decision.
- infinite-dimensional fixed point problem

Krusell-Smith Algorithm

KS Algorithm: Approximate the distribution with a small number of moments (often mean and variance).

- if future prices are accurately forecasted by the small number of moments: globally accurate and can capture the global non-linearities.
- if the low-order moments cannot fully capture the price dynamics, i.e. when firms follow (S,s) rule, KS algorithm, or "Approximate Aggregation" fails.
- need "Explicit Aggregation" (XPA, DR2010-JEDC) or perturbation and projection (Reiter 2009, Winberry 2018).

Equilibrium

The equilibrium in general features two parts:

- policy rule for control variables
- law of motion of state variables

In RA models:

- individual policy rule $=$ aggregate policy rule
- LM of individual state variables$=$ ALM

Not true for HA models:

- individual policy rule $\rightarrow$ aggregation w. distn $\rightarrow$ aggregate policy rule
- LM of individual state variables$\rightarrow$ aggregation w. distn $\rightarrow$ ALM

Individual Problem: Grids

Individual Problem:

$$\tilde{k}' = [(1-\tau_t)\varepsilon + \mu(1-\varepsilon)]w + (1-\delta+r)k -$$

$$\{h + \beta E[\frac{1-\delta+r'}{[(1-\tau')w'\varepsilon' + \mu(1-\varepsilon')w' + (1-\delta+r')k' - k'(k')]^{\gamma}}]\}^{-1/\gamma}$$

We solve this equation following an iterative procedures on a grid.

Grid of points: (k, ε, m, a).
Restrictions on the grid: $k \in [0, k_{max}]$; $m \in [m_{min}, m_{max}]$.
Similar to KS(1998), we assume first moment is sufficient.
Grid of points: (k, ε, Kmean, a).
Restrictions: $k \in [0, k_{max}]$; $Kmean \in [Kmean_{min}, Kmean_{max}]$.

Individual Problem: Iterative Procedures

Individual Problem:

$$\tilde{k}' = [(1-\tau_t)\varepsilon + \mu(1-\varepsilon)]w + (1-\delta+r)k -$$

$$\{h + \beta E[\frac{1-\delta+r'}{[(1-\tau')w'\varepsilon' + \mu(1-\varepsilon')w' + (1-\delta+r')k' - k'(k')]^{\gamma}}]\}^{-1/\gamma}$$

We solve this equation following an iterative procedures on a grid.

Given initial states a and ε_i for all i, r and w (on RHS) are known.
Initial capital function: k'(k, ε, Kmean, a)=0.9k.
k' is known, thus K' and $E(r')$ (on RHS) are known.
With transition probabilities, $E(\tau')$, $E(w')$ and $E(\varepsilon')$ are known.
set h=0.
New capital function $\tilde{k}'$ is known for any k.
Updated capital function: $\tilde{\tilde{k}}' = \eta\tilde{k}'() + (1-\eta)k'()$.

Individual Problem: Practical Issues

- k_{max}. We can set k_{max} very large: all k' fall into $[0, k_{max}]$, but it's very costly in computation.
  We instead set a relatively large k_{max}, and bound k' whenever it exceeds the grid. (in our case we set $k_{max} = 1000$)
- Occasionally binding constraint. We need more grid points at low level of capital and fewer points at high level of capital.
  A simple polynomial rule for placement of grid points:

$$k_j = (\frac{j}{J})^{\theta} k_{max}, \quad j = 0, 1, 2, ..., J \tag{8}$$

  $\theta = 1$: equal distance b/w grid points;
  $\theta > 1$: concentration at the bottom.
- updating parameter (η): trade-off b/w speed and stability.
- convergence parameter: time to stop.

Aggregate Problem: ALM

- We *approximate* aggregate law of motion by:

$$m' = f(m, a; b) \tag{9}$$

where b is a vector of ALM coefficient (this is regression!).

- We estimate the following equations in two aggregate states:

$$log(K_{t+1}) = b_1 + b_2 log(K_t), \text{if state is good;} \tag{10}$$

$$log(K_{t+1}) = b_3 + b_4 log(K_t), \text{if state is bad;} \tag{11}$$

- Stochastic Simulation: This paper
- Non-stochastic Simulation: Young (2010 JEDC); Den Haan (2010 JEDC)

Aggregate Problem: Iterative Procedures

- Fixed initial capital distribution, initial aggregate shocks and initial idiosyncratic shocks. (N=10,000)
- Generate time series of T period aggregate shocks, and idiosyncratic shocks.
- Guess an initial vector of coefficients b. (i.e., [0,1;0,1]:

$$log(K_{t+1}) = 0 + log(K_t), \text{if state is good or bad;}$$

- Solve the *Individual Problem*.
- Simulate the economy for T periods forward, explicitly solve cross-sectional capital holding, and calculate the mean K_t.
- Regress K_{t+1} on K_t[1], get new vector of coefficients $\tilde{b}$.
- Updated vector of coefficients: $\tilde{\tilde{b}} = \lambda \tilde{b} + (1 - \lambda) b$.

[1] discard 100 initial periods to mitigate the effect of initial distribution

program

The program includes the following subroutines:

- "MAIN.m" (computes a solution and stores the results in "Solution")
- "SHOCKS.m" (a subroutine of MAIN.m; generates the shocks)
- "INDIVIDUAL.m" (a subroutine of MAIN.m; computes a solution to the individual problem)
- "AGGREGATE.m" (a subroutine of MAIN.m; performs the stochastic simulation)
- "Inputs_for_test" (contains initial distribution of capital and 10,000-period realizations of aggregate shock and idiosyncratic shock for one agent provided by Den Haan, Judd and Juillard, 2008)

program: MAIN.m

MAIN.m include the following sections:

- parameters: including model parameters, stimulation parameters,transition probabilities, steady state values of capital
- shocks: call "SHOCK.m" functions for aggr. and idio. shocks.
- grids: including capital, moments of capital (mean)
- initials: including capital evolution function, distribution, ALM
- convergence: including initial diff value, criteria, updating parameters)
- solver: call "INDIVIDUAL.m" and "AGGREGATE.m" functions
- figures

program: SHOCK.m

T periods and N agents

- aggregate shocks: (T,1);
- idiosyncratic shocks: (T,N)
- given an initial agg. state
- generate cross-sectional initial idios. state accordingly
- simulate agg. shocks T periods forward with transition prob
- simulate idios. shocks T periods forward with transition prob, conditional on evolution of aggregate states

program: INDIVIDUAL.m

Iterative procedures:

- auxilary matrices of transition prob on the grid
- auxilary matrices of k, Kmean, a, e on the grid
- r, w and wealth(t)
- c and u'(c)
- Kmean'
- r', w' and wealth(t+1)
- c' and u'(c')
- update k'
- update c

Comments [2]

Advantage of KS algorithm:

- simple and intuitive
- widely used

Disadvantage of KS algorithm:

- *approximate* aggregate
- can the distribution be summarized by mean and variance?
- sampling noise in simulation
- computatonal cost

[2] see Den Haan(2010) for a discussion on KS algorithm

Reference and Further Reading

Reference

- Maliar, L., Maliar, S., Valli, F. (2010). Solving the incomplete markets model with aggregate uncertainty using the Krusell–Smith algorithm. Journal of Economic Dynamics and Control, 34(1), 42-49.

Further Reading for non-stochastic simulation method

- Den Haan, W. J. (2010). Comparison of solutions to the incomplete markets model with aggregate uncertainty. Journal of Economic Dynamics and Control, 34(1), 4-27.
- Young, E. R. (2010). Solving the incomplete markets model with aggregate uncertainty using the Krusell–Smith algorithm and non-stochastic simulations. Journal of Economic Dynamics and Control, 34(1), 36-41.

Reference and Further Reading

Further Reading for KS Algorithm/Application

- (classic) Krusell, P., Smith, Jr, A. A. (1998). Income and wealth heterogeneity in the macroeconomy. Journal of political Economy, 106(5), 867-896.
- (review) Terry, S. J. (2017). Alternative methods for solving heterogeneous firm models. Journal of Money, Credit and Banking, 49(6), 1081-1111.
- (application) Khan, A., Thomas, J. K. (2008). Idiosyncratic shocks and the role of nonconvexities in plant and aggregate investment dynamics. Econometrica, 76(2), 395-436.