Week 3: Stationary Equilibrium of HA Model

Computation Study Group

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Special thanks to Prof.Jinhui Bai!

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Aiyagari (1994) Model

- A household saving problem

$$V(k,\epsilon) = \max_{c,a'} \left\{ \frac{c^{1-\sigma}}{1-\sigma} + \beta \, \mathrm{EV}\left(k',\epsilon'\right) \right\}$$

subject to

$$c + k' = (1 + r - \delta)k + w\epsilon\bar{l}$$
$$c \geq 0, k' \geq -\phi$$

ϵ is idiosyncratic labor productivity shock.

- Firm's problem

$$\max = K^{\alpha} N^{1-\alpha} - rK - wN$$

For now, no aggregate TFP shock.

Stationary Recursive Competitive Equilibrium

A stationary recursive competitive equilibrium is a set of functions, $v(k,\epsilon)$ and $g(k,\epsilon)$, a set of prices and quantities (r,w,K,N), and a stationary distribution $\lambda(k,\epsilon)$ such that

- Given (r,w), $v(k,\epsilon)$ and $g(k,\epsilon)$ solve the household's dynamic programming problem.
- Prices are competitively determined:

$$w = (1-\alpha)\left(\frac{K}{N}\right)^{\alpha}, \quad r = \alpha\left(\frac{K}{N}\right)^{\alpha-1} - \delta$$

- Market clears:

$$K = \sum_{\epsilon}\sum_{k}\lambda(k,\epsilon)g(k,\epsilon), \quad N = \sum_{\epsilon}\sum_{k}\lambda(k,\epsilon)\epsilon\bar{l}$$

- $\lambda(k,\epsilon)$ is a stationary distribution from $g(k,\epsilon)$.

Some Math Preparation

Some Key Elements in Numerical Computation

- Discretization
- Function Approximation
- Optimization
- Root Finding / Equation Solving

Function Approximation

How to approximate a continuous function from discrete function values?

- We can use piece-wise polynomial approximation
- Idea: Construct a low-order polynomial for every two neighboring grid points.
- We introduce two methods
  - Cubic Spline
  - Piecewise Cubic Hermite Interpolation Polynomial (PCHIP)

Function Approximation

Cubic Spline: MATLAB function "spline"

- A Cubic Spline is a set of piecewise cubic polynomials $\widehat{f}(x)$ for each $n = 1, 2, \ldots, N-1$ and $x \in [x_n, x_{n+1}]$

$$\widehat{f}_n(x) = c_{n0} + c_{n1}(x - x_n) + c_{n2}(x - x_n)^2 + c_{n3}(x - x_n)^3$$

such that

  - Function value is continuous for all nodes: $\widehat{f}_n(x_n) = y_n$ and $\widehat{f}_{n+1}(x_{n+1}) = y_{n+1}$ for all $n = 1, 2, \ldots, N-1$
  - First-order derivative is continuous for each interior node: $\widehat{f}'_n(x_n) = \widehat{f}'_{n+1}(x_n)$ for $2 \leq n \leq N-1$
  - Second-order derivative is continuous for each interior node: $\widehat{f}''_n(x_n) = \widehat{f}''_{n+1}(x_n)$ for $2 \leq n \leq N-1$

- How can we pin down the coefficients?
  - We have $4(N-1)$ unknown parameters, but only $2(N-1) + 2(N-2) = 4N - 6$ restrictions.
  - Need two more conditions, for example
    - "not-a-knot": $\widehat{f}'''_1(x_2) = \widehat{f}'''_2(x_2)$, $\widehat{f}'''_{N-2}(x_{N-1}) = \widehat{f}'''_{N-1}(x_{N-1})$
    - Requirements on $\widehat{f}'_1(x_1)$ and $\widehat{f}'_{N-1}(x_N)$.

Function Approximation

Piecewise Cubic Hermite Interpolation Polynomial: MATLAB function "pchip"

- Suppose on each node, we have data on both function value and first derivative value: $(x_n, y_n, y'_n)_{n=1}^N$, where

$$y_n = f(x_n)$$
$$y'_n = f'(x_n)$$

- Then on each interval $[x_n, x_{n+1}]$, the data uniquely determines a cubic polynomial

$$\widehat{f}_n(x) = c_{n0} + c_{n1}(x - x_n) + c_{n2}(x - x_n)^2 + c_{n3}(x - x_n)^3$$

for $x \in [x_n, x_{n+1}]$ through four conditions:

$$y_n = \widehat{f}_n(x_n), y_{n+1} = \widehat{f}_{n+1}(x_{n+1}),$$
$$y'_n = f\widehat{f}'_n(x_n), y'_{n+1} = \widehat{f}'_n(x_{n+1}).$$

- In reality, we usually don't have data on derivatives. MATLAB function "pchip" approximate it by average of two slopes.

Function Approximation

Comparison between Interpolation Methods

- Cubic spline is more smooth. We can easily calculate first and second order derivatives from it.
- PCHIP is more shape-preserving. It can better preserve the shape of a kinked line (for example, the policy function in the Aiyagari model).

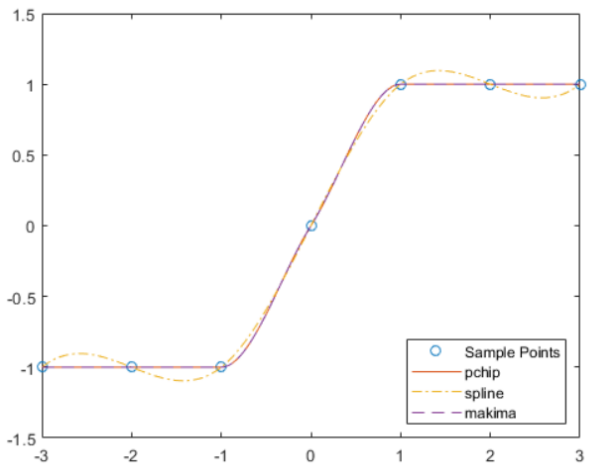


Figure: Comparison between Interpolation Methods

Root Finding

How to find root(s) for a non-linear equation $f(x) = 0$?

- Bracketing Method
  - Step 1: Find an interval (bracket) (a, b) such that $f(a)f(b) < 0$.
  - Step 2: Find a point x inside the bracket. If $f(a)f(x) > 0$, let $a = x$; if $f(b)f(x) > 0$, let $b = x$
  - Step 3: Redo Step 2 on new (a, b)
  - Step 4: Break when $|b - a|$ is sufficiently small. Then x is the root we find.

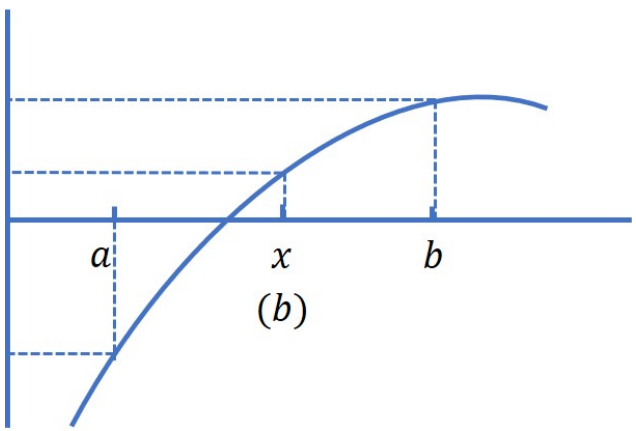


Figure: Bracketing Method

Root Finding

Now the question is: how to find such a x inside the bracket (a, b)?

- A naive way: bisection.
- More efficient way: by linear approximation.
  In Step k, approximate $f(x)$ around last Step's x_{k-1}:

$$f(x) \approx f(x_{k-1}) + A_k(x - x_{k-1})$$

$$f(x_{k-1}) + A_k(x - x_{k-1}) = 0 \Rightarrow x_k = x_{k-1} - A_k^{-1} f(x_{k-1})$$

- How to choose A_k?
  - Fixed point iteration: A_k =1.
  - Newton's method: $A_k = f'(x_{k-1})$.

Root Finding

- Fixed point iteration

$$x_k = x_{k-1} - f(x_{k-1})$$

- Newton's method

$$x_k = x_{k-1} - f(x_{k-1})^{-1} f(x_{k-1})$$

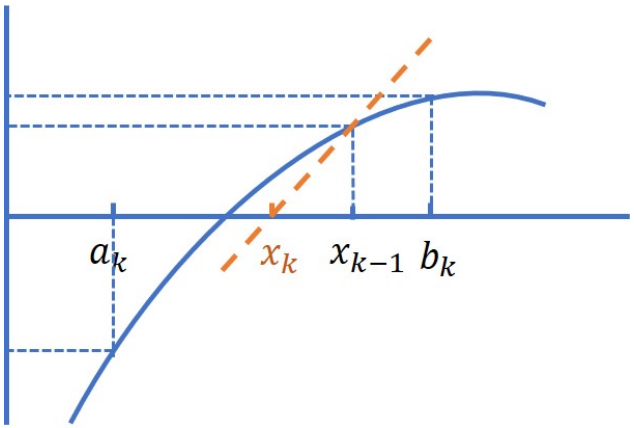


Figure: Fixed Point Iteration

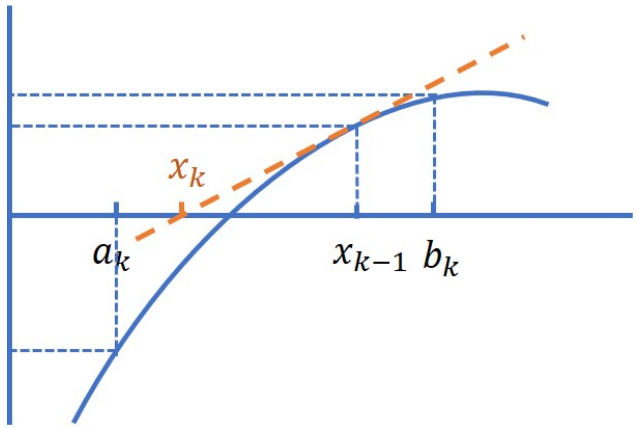


Figure: Newton's Method

Root Finding

MATLAB built-in functions for equation solving

- fzero: solves one-dimensional non-linear equation
- fsolve: solves multi-dimensional non-linear equations
- Note: The idea of N-D non-linear equation solving is different from 1-D case: it actually tries to solve the global minimum of a quadratic function and uses function optimization. Hence, directly uses optimization algorithm if you can.
- Recommend you to read MATLAB documentation.

Optimization

How to find local minimum for a function $f(x)$?
Idea:

- We still use Bracketing Method: shrink bracket $[a, b]$ until we find a local minimum.
- A simple way: Bisection section search.
- More efficient way: by quadratic approximation.

$$\widehat{f}(x) = c_0 + c_1 x + c_2 x^2$$

If $c_2 > 0$, a candidate iteration point is given by the minimizer

$$\arg\min \widehat{f}(x) = -\frac{c_1}{2c_2}$$

If $c_2 < 0$ or $\arg\min \widehat{f}(x) \notin [a, b]$, update by safe methods like bisection search.

Optimization

How to solve coefficient c_0, c_1 and c_2 in $\widehat{f}(x)$?

- Brent's Method: Use three function values.

  MATLAB function: fminbnd

- Quasi-Newton Method: Use one function value and two first derivatives.

$$\widehat{f}(x) = f\left(x^{(k)}\right) + f'\left(x^{(k)}\right)\left(x - x^{(k)}\right) + \frac{1}{2}A^{(k)}\left(x - x^{(k)}\right)^2$$

  where

$$A^{(k)} = \frac{f'\left(x^{(k)}\right) - f'\left(x^{(k-1)}\right)}{x^{(k)} - x^{(k-1)}}$$

  MATLAB function: fmincon

- Newton Method: Use one function value, one first derivative and one second derivative.

$$\widehat{f}(x) = f\left(x^{(k)}\right) + f'\left(x^{(k)}\right)\left(x - x^{(k)}\right) + \frac{1}{2}f''\left(x^{(k)}\right)\left(x - x^{(k)}\right)^2$$

  MATLAB function: fmincon

Individual household's dynamic programming problem

Solution Methods: An Overview

- Bellman Equation Methods
  - Value function iteration
  - Value function iteration with Howard improvement
- Euler Equation Methods
  - Euler equation iteration / Policy function iteration
  - Euler equation perturbation method (Dynare)

Bellman Equation Methods

- Bellman Equation:

$$V(k,\epsilon) = \max_{k'} \left\{ \frac{((1+r-\delta)k + w\epsilon - k')^{1-\sigma}}{1-\sigma} + \beta EV\left(k', \epsilon'\right) \right\}$$

subject to

$$-\phi \leq k' \leq (1+r-\delta)k + w\epsilon$$

- Our goal:
Solve value function $V(k,\epsilon)$ and policy function $k' = G(k,\epsilon)$.

Discretization of State Variables

We discretize the domain of functions $V(k, s)$ and $G(a, s)$. That is, we discretize state variables k and s.

- Discretization of k : k and k' lies on a N by 1 grid with

$$n \in \mathcal{N} = \{1, 2, \ldots, N\}$$
$$k \in \mathcal{K} = \{k_1, k_2, \ldots, k_N\}$$

- Discretization of ϵ: ϵ follows S-state Markov Chain with state space

$$s \in \mathcal{S} = \{1, 2, \ldots, S\}$$
$$\epsilon \in \mathcal{E} = \{\epsilon_1, \epsilon_2, \ldots, \epsilon_S\}$$

and a S by S Transition Probability Matrix $\mathcal{P}$

$$\mathcal{P}\left(s, s'\right) = \Pr\left(\epsilon_{t+1} = \epsilon_{s'} \mid \epsilon_t = \epsilon_s\right)$$

Note: methods to discretize an AR(1) process into $\mathcal{P}$
(1) Rouwenhorst (1995); (2) Tauchen (1991).

Discretization of State Variables

Now the Bellman Equation becomes

$$V(k_n, \epsilon_s) = \max_{k'} \left\{ \frac{\left((1 + r - \delta)k_n + w\epsilon_s - k'\right)^{1-\sigma}}{1 - \sigma} + \beta \sum_{s'=1}^{S} \mathcal{P}(s, s') V\left(k', \epsilon_{s'}\right) \right\}$$

subject to

$$-\phi \leq k' \leq (1 + r - \delta)k_n + w\epsilon_s$$

$$k' \in \mathcal{K} = \{k_1, k_2, \ldots, k_N\}$$

- Our goal:
  Solve value function $V(k_n, \epsilon_s)$ and policy function $k' = G(k_n, \epsilon_s)$ for

$$n \in \mathcal{N} = \{1, 2, \ldots, N\}$$

$$s \in \mathcal{S} = \{1, 2, \ldots, S\}$$

Value Function Iteration: Idea

- We are essentially solving a root finding problem:

$$V = TV$$

$$f(V) = V - TV = 0$$

- We can solve it by fixed point iteration
  - Step 0: Choose an initial value function V.
  - Step 1: Obtain new value function V' by

$$V' = V - f(V) = V - (V - TV) = TV$$

  - Step 2: Check if $||V' - V|| < t_v$, where t_v is a predetermined tolerance level. If not, let $V = V'$, and redo Step 1-2. Break if $||V' - V|| < t_v$ or number of iteration $> MaxIter_v$
- It is value function iteration.

Value Function Iteration

Then how can we perform this iteration?
Here we use a continuous-state method, by using function interpolation.

- Begin with old value function
  $V(k_n, \epsilon_s)$ $(n \in \mathcal{N} = \{1, 2, \ldots, N\},\ s \in \mathcal{S} = \{1, 2, \ldots, S\})$
- Our goal: obtain new value function on each grid point (k_n, ϵ_s).
- Interpolation
  - Purpose:
    $V(k_n, \epsilon_s),\ n \in \mathcal{N} = \{1, 2, \ldots, N\}, s \in \mathcal{S} = \{1, 2, \ldots, S\} \rightarrow$
    $V(k, \epsilon_s),\ k \in [k_1, k_N], s \in \mathcal{S} = \{1, 2, \ldots, S\}$
  - Cubic spline: vfn = spline(agrid,v.');
  - Evaluation: ppval(vfn,aprime);

Value Function Iteration

- Maximization

$$V(k_n, \epsilon_s) = \max_{k'} \left\{ \frac{((1 + r - \delta)k_n + w\epsilon_s - k')^{1-\sigma}}{1 - \sigma} + \beta \sum_{s'=1}^{S} \mathcal{P}(s, s') V\left(k', \epsilon_{s'}\right) \right\}$$

subject to

$$k' \in [\max\{-\phi, k_1\}, \min\{(1 + r - \delta)k_n + w\epsilon_s, k_N\}]$$

  - Constrained Optimization
    MATLAB built-in functions: fminbnd, fmincon.
  - We obtain
    (1) new value function $V'(k_n, \epsilon_s)$
    (2) policy function $k' = g(k_n, \epsilon_s)$

Howard Improvement: Idea

- In value function iteration, we have a byproduct: policy function $k' = g(k_n, \epsilon_s)$.
- But in previous value function iteration, we completely ignore the information in g.
- Now, how about utilizing the information in g ?
  An idea: if $g(k_n, \epsilon_s)$ is the true policy function, then we have

$$V(k_n, \epsilon_s) = \frac{\left((1 + r - \delta)k_n + w\epsilon_s - k'\right)^{1-\sigma}}{1-\sigma} + \beta \sum_{s'=1}^{S} \mathcal{P}(s, s') V\left(k', \epsilon_{s'}\right)$$

  Then, given
  $k' = g(k_n, \epsilon_s), (n \in \mathcal{N} = \{1, 2, \ldots, N\},\ s \in \mathcal{S} = \{1, 2, \ldots, S\})$,
  we can solve for $V(k_n, \epsilon_s)$.

Howard Improvement

But how can we solve for $V(k_n, \epsilon_s)$?
Again, it is an equation solving problem – we can use fixed point iteration!

- Step 0: Choose an initial value function $V(k_n, \epsilon_s)$.
- Step 1: Obtain a new value function $V'(k_n, \epsilon_s)$ by

$$V'(k_n, \epsilon_s) = \frac{((1 + r - \delta)k_n + w\epsilon_s - k')^{1-\sigma}}{1 - \sigma} + \beta \sum_{s'=1}^{S} \mathcal{P}(s, s') V(k', \epsilon_{s'})$$

- Step 2: Check if $||V' - V|| < t_h$. If not, let $V = V'$ and redo Step 1-2.
  Break if $||V' - V|| < t_h$ or number of iteration $> MaxIter_h$

Value Function Iteration + Howard Improvement

Now we combine VFI and Howard Improvement.

- Step 0: Initialization
  (1) Set initial value function $V^0(k_n, \epsilon_s)$ and policy function $G^0(k_n, \epsilon_s)$.

  (2) Set tolerance level for value function, policy function and Howard Improvement step: t_v, t_p, t_{hp}, and t_h.

  (3) Set maximum iteration number K_v, and J_h.

- Step 1: Value function iteration.

  In iteration $k = 1, ..., K_v$, use continuous state VFI to calculate value function $\hat{V}^k$ and policy function G^k.

- Step 2: Check. If $||V^k - V^{k-1}|| < t_v$ and $||G^k - G^{k-1}|| < t_p$, declare success with the solution $V = V^{k-1}$ and $G = G^k$. Otherwise, go to Step 3.

Value Function Iteration + Howard Improvement

- Step 3: Update.

  If $||G^k - G^{k-1}|| < t_{hp}$, then update V^k by Howard Improvement.

  - Step 3.0 Let $\hat{V}^k$ be initial value in Howard Improvement: $V_h^0 = \hat{V}^k$.
  - Step 3.1 For iteration $j = 1, ..., J_h$,

$$V_h^j(k_n, \epsilon_s) = \frac{\left((1 + r - \delta)k_n + w\epsilon_s - G^k(k_n, \epsilon_s)\right)^{1-\sigma}}{1 - \sigma}$$
$$+\beta \sum_{s'=1}^{S} \mathcal{P}(s, s') V_h^{j-1}\left(G^k(k_n, \epsilon_s), \epsilon_{s'}\right)$$

  - Step 3.2 Check: if $||V^j - V^{j-1}|| < t_v$, break; otherwise, back to Step 3.1.

  Update V^k by $V^k = V_h$, which is obtained in the Howard Improvement process.

  If $||G^k - G^{k-1}|| \geq t_{hp}$, then update V^k by original VFI value: $V^k = \hat{V}^k$.

Euler Equation Methods

Euler equation in the Aiyagari Model (suppose interior solution)

$$c^{-\sigma} = \beta(1+r)Ec'^{-\sigma}$$

$$((1+r-\delta)k + w\epsilon - k')^{-\sigma} = \beta(1+r)E((1+r-\delta)k' + w\epsilon' - k'')^{-\sigma}$$

$$((1+r-\delta)k + w\epsilon - g(k,\epsilon))^{-\sigma} = \beta(1+r)E((1+r-\delta)g(k,\epsilon) + w\epsilon' - g(k',\epsilon'))^{-\sigma}$$

where g is the policy function: $k' = g(k,\epsilon)$ and $k'' = g(k',\epsilon')$.

- Euler equation gives a functional equation of policy function g: again, an equation-solving problem.
- Again, we can use fixed point iteration.
  Given a policy function $\tilde{g}$, we can solve for a new policy function g by solving the root of equation

  $$((1+r-\delta)k+w\epsilon-g(k,\epsilon))^{-\sigma} = \beta(1+r)E((1+r-\delta)g(k,\epsilon)+w\epsilon'-\tilde{g}(k',\epsilon'))^{-\sigma}$$

  Iterate until $||\tilde{g} - g|| < t_p$.

Policy Function Iteration

Recall that in practice, policy function is on discrete grids:

$$k' = g(k_n, \epsilon_s), (n \in \mathcal{N} = \{1, 2, \ldots, N\},\ s \in \mathcal{S} = \{1, 2, \ldots, S\})$$

Then there are two types on policy function iteration methods:

- Exogenous Grid Method
- Endogenous Grid Method

Policy Function Iteration: Exogenous Grid Method

Euler Equation:

$$((1+r-\delta)k_n+w\epsilon_s-k')^{-\sigma} = \beta(1+r)E((1+r-\delta)k'+w\epsilon_{s'}-g(k',\epsilon_{s'}))^{-\sigma}$$

- Step 0: Choose an initial policy function $g(k_n, \epsilon_{s'})$ $(n \in \mathcal{N} = \{1, 2, \ldots, N\},\ s \in \mathcal{S} = \{1, 2, \ldots, S\})$.
- Step 1: Use interpolation to approximate continuous policy functions $\tilde{g}(k', \epsilon_{s'})$, $s \in \mathcal{S} = \{1, 2, \ldots, S\}$.
- Step 2: For each n and s, solve new policy function $k' = g'(k_n, \epsilon_s)$ from Euler Equation.
- Iterate until the convergent of policy function g on grid points.

Step 2 is time-consuming, since it involves solving a non-linear equation.

Policy Function Iteration: Endogenous Grid Method

Euler Equation:

$$((1+r-\delta)k_n+w\epsilon_s-k')^{-\sigma}=\beta(1+r)E((1+r-\delta)k'+w\epsilon_{s'}-g(k',\epsilon_{s'}))^{-\sigma}$$

- Step 0: Choose an initial policy function $g(k_n,\epsilon_{s'})$.
- Step 1: Endogenous Grid.
  For each today's $\epsilon=\epsilon_s$ and each future $k'=k_{n'}$ and $k''=g\left(k_{n'},\epsilon_{s'}\right)$, solve today's k from Euler Equation:

  $$\widehat{k}_{n's}=\frac{RHS^{-\frac{1}{\sigma}}+k_{n'}-w\epsilon_s}{1+r-\delta},\ RHS=\beta(1+r)E((1+r-\delta)k_{n'}+w\epsilon_{s'}-g(k_{n'},\epsilon_{s'}))^{-\sigma}$$

- Step 2: Function Approximation and Interpolation
  For each today's $\epsilon=\epsilon_s$, now we have $\left(\widehat{k}_{n's},k_{n'}\right)_{n'=1}^{N}$. Use interpolation to obtain a continuous policy function $\tilde{g}\left(k,\epsilon_s\right)$. Evaluate $\tilde{g}$ at exogenous grid point $\{k_1,k_2,\ldots,k_N\}$ to get new policy function $g'(k_n,\epsilon_s)$.
- Iterate until the convergent of policy function g on grid points.

Endogenous Grid Method: Corner Solutions

Considering the possibility of corner solutions, Euler Equation becomes

$$((1+r-\delta)k_n+w\epsilon_s-k')^{-\sigma} \geq \beta(1+r)E((1+r-\delta)k'+w\epsilon_{s'}-g(k',\epsilon_{s'}))^{-\sigma}$$

"$>$" implies $k'=0$ while $k'>0$ implies "$=$".
How to deal with it?
– Add & Drop

Endogenous Grid Method: Corner Solutions

If $\widehat{k}_{n's} < \phi$, discard $\left(\widehat{k}_{n's}, k_{n'}\right)$ pair. For $k' = k_1 = \phi$, add all grid point pair (k_n, ϕ) to endogenous grids, where $k_n \leq \widehat{k}_{1s}$.

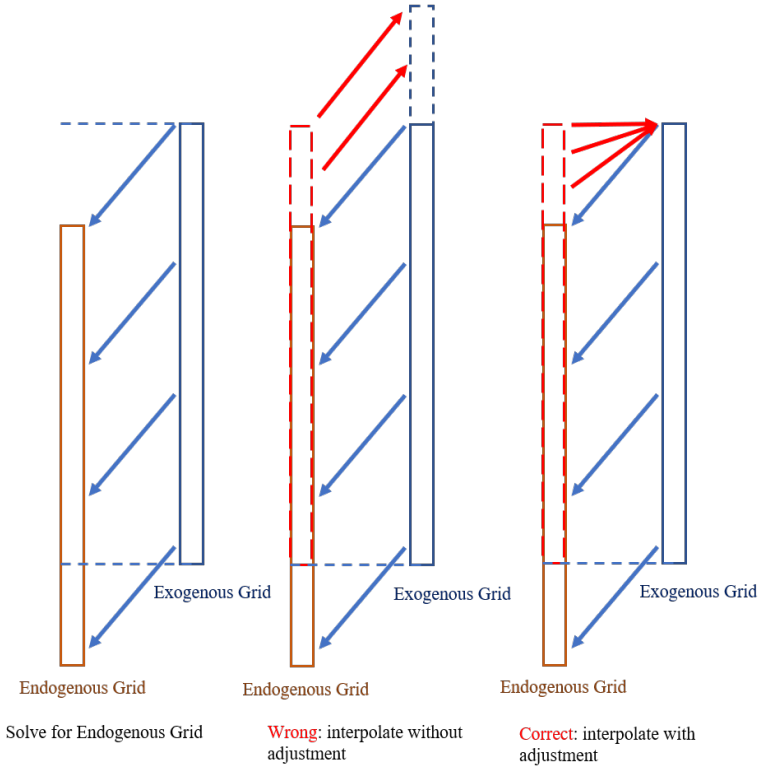


Figure: "Add" Case

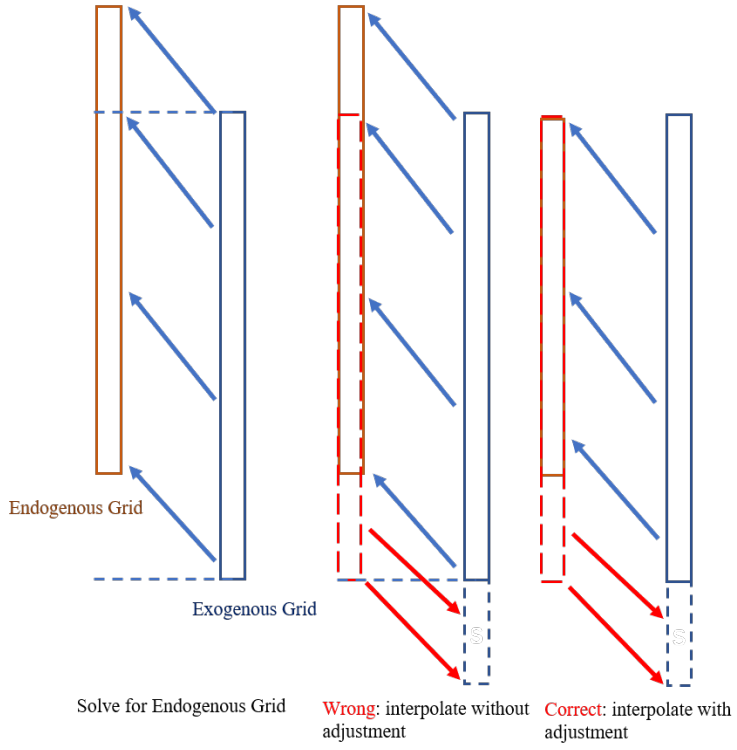
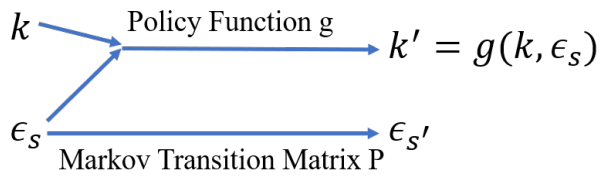


Figure: "Drop" Case

Stationary Distribution

Evolution of Probability Distribution

- Evolution of an individual's state (k, ϵ_s)



- With a continuous k, (k, ϵ_s) follows a continuous-state Markov process with transition prob density function given by

$$Q\left(\left(k, \epsilon_s\right), \left(k', \epsilon_{s'}\right)\right) = \mathcal{P}\left(s, s'\right) \cdot \mathcal{I}\left(k' = g\left(k, \epsilon_s\right)\right)$$

- Evolution of the distribution
  The distribution over (k, ϵ_s), $\lambda(k, \epsilon_s)$, evolves according to

$$\lambda_{t+1}\left(k', \epsilon_{s'}\right) = \sum_s \int Q\left(\left(k, \epsilon_s\right), \left(k', \epsilon_{s'}\right)\right) d\lambda_t(k, \epsilon_s),$$

- Stationary distribution is defined as $\lambda(k, \epsilon_s)$ such that

$$\lambda\left(k', \epsilon_{s'}\right) = \sum_s \int Q\left(\left(k, \epsilon_s\right), \left(k', \epsilon_{s'}\right)\right) d\lambda(k, \epsilon_s)$$

Calculation of Stationary Distribution

Stationary distribution

$$\lambda\left(k', \epsilon_{s'}\right) = \sum_{s} \int Q\left(\left(k, \epsilon_s\right), \left(k', \epsilon_{s'}\right)\right) d\lambda(k, \epsilon_s)$$

- Our goal is to numerically calculate the stationary distribution.
- Generally, there are two methods.
- Discretization Method
  Approximate transition probability density function $Q((k, \epsilon_s), (k', \epsilon_{s'}))$ by a Markov transition matrix Q.
  Then we can calculate stationary distribution by this Markov transition matrix Q.
- Stochastic Simulation Method
  Simulates a large number of households over a long period of time. Then we can finally obtain the stationary distribution.

Stationary Distribution: Discretization Method

Idea:

- First, imagine an ideal case: policy function $k' = g(k_n, \epsilon_s)$ happens to lie on the girds $\mathcal{K} = \{k_1, k_2, \ldots, k_N\}$.
  That is, for any k', there exists a $n' \in \mathcal{N} = \{1, 2, \ldots, N\}$, such that $k' = k_{n'}$.
- Then things become easy. Q becomes a $NS \times NS$ transition matrix:

$$Q\left((n,s),(n',s')\right) = \begin{cases} \mathcal{P}(s,s') & \text{if } n' = g(n,s) \\ 0 & \text{if } n' \neq g(n,s) \end{cases}$$

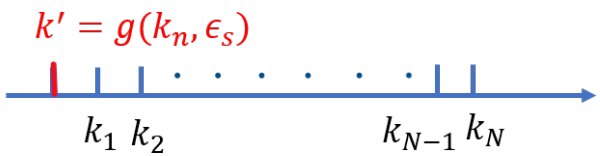
- But we know in reality, it is almost impossible that k' exactly lies on the grid points.
- Then, one feasible way is that we assign probability values to gird points based on their distance to k'.

Stationary Distribution: Discretization Method

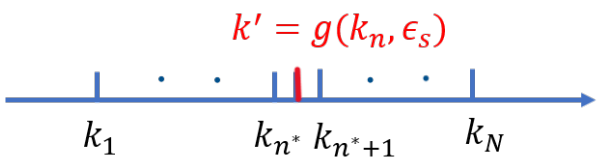
Eric Young's Method (2010, JEDC) to obtain a $NS \times NS$ transition matrix Q

- For each (k_n, ϵ_s), we can calculate $Q((n,s),(n',s'))$ $(n' \in \mathcal{N} = \{1,2,\ldots,N\},\ s' \in \mathcal{S} = \{1,2,\ldots,S\})$ by the following way:

$$Q((n,s),(n',s')) = \begin{cases} 0, n' = 1,\ldots,N-1 \\ P(s,s'), \quad n' = N \end{cases}$$



$$Q((n,s),(n',s')) = \begin{cases} 0, n' = 2,\ldots,N \\ P(s,s'), n' = 1 \end{cases}$$



$$Q((n,s),(n',s')) = \begin{cases} 0, n' = 1,\ldots,n^*-1 \\ \dfrac{k_{n^*+1} - k'}{k_{n^*+1} - k_{n^*}} P(s,s'), n' = n^* \\ \dfrac{k' - k_{n^*}}{k_{n^*+1} - k_{n^*}} P(s,s'), n' = n^*+1 \\ 0, n' = n^*+2,\ldots,N \end{cases}$$

Stationary Distribution: Discretization Method

Calculate stationary distribution from transition matrix Q.

- Probability Evolution

$$\lambda_t = \lambda_t(k_n, \epsilon_s)$$

$$\lambda_{t+1} = Q^T \lambda_t$$

- Stationary Distribution

$$\lambda = Q^T \lambda$$

- Two methods
  - Method of eigenvalue and eigenvector
    λ is the eigenvector which corresponds to eigen value 1 of matrix Q^T.
  - Iteration
    Again, it is a equation solving problem. Just use fixed point iteration.

Stationary Distribution: Stochastic Simulation

- Step 0

  Fix I agents, T periods, and an initial distribution $\left(k_0^i, s_0^i\right)_{i=1}^I$.

- Step 1

  In $0 \leq t \leq T-1$, use the policy function $k' = g(k, s)$ to calculate $\left(k_{t+1}^i\right)_{i=1}^I$ for each $i \in I$, i.e.

  $$k_{t+1}^i = g\left(k_t^i, s_t^i\right)$$

  and use transition matrix $\mathcal{P}\left(s, s'\right)$ of shock s and a random number generator to generate $\left(s_{t+1}^i\right)_{i=1}^I$

- Step 2

  Collect the simulated panel data with $(T+1)$ periods and I households, $\left(k_t^i, s_t^i\right)_{i=1,t=0}^{I,T}$.

- Step 3

  If the change in distributions is small between $T-1$ and T, stop. Otherwise, pick a larger T and go back to Step 0.

Solve Equilibrium

Capital Market Clearing Condition

- Capital demand from firms: K
- Capital supply from household:

$$\sum_{s=1}^{S}\sum_{n=1}^{N}\lambda(k_n,\epsilon_s)g(k_n,\epsilon_s)$$

or equivalently

$$\sum_{n=1}^{N}\left(\sum_{s=1}^{S}\lambda(k_n,\epsilon_s)\right)k_n$$

- Market clears:

$$K=\sum_{s=1}^{S}\sum_{n=1}^{N}\lambda(k_n,\epsilon_s)g(k_n,\epsilon_s)=\sum_{n=1}^{N}\left(\sum_{s=1}^{S}\lambda(k_n,\epsilon_s)\right)k_n$$

Equilibrium Conditions

Recall our equilibrium conditions.

- Given (K, N), (w, r) is determined competitively by

$$w = (1-\alpha)\left(\frac{K}{N}\right)^{\alpha}, \quad r = \alpha\left(\frac{K}{N}\right)^{\alpha-1} - \delta$$

- Given (r, w), $g(k, \epsilon)$ is the policy function from household's dynamic programming problem.
- Given policy function $g(k, \epsilon)$ and transition matrix P, $\lambda(k, \epsilon)$ is the stationary distribution.
- Market clearing condition for K and N:

$$K = \sum_{s=1}^{S}\sum_{n=1}^{N}\lambda(k_n, \epsilon_s)g(k_n, \epsilon_s), \quad N = \sum_{s=1}^{S}\sum_{n=1}^{N}\lambda(k, \epsilon_s)\epsilon_s\bar{l} = \sum_{s=1}^{S}\mu(\epsilon_s)\epsilon_s\bar{l}$$

where μ is the invariant distribution of labor productivity shock, given by $P^{-1}\mu = \mu$.

Equilibrium Conditions

We know in this model, N is exogenously determined by P and $\bar{l}$. Then equilibrium conditions can be summarized as a equation of K: $f(K) = 0$, where function value $f(K)$ is defined by the following procedure.

- Step 1: Given $N = \sum_{s=1}^{S} \mu(\epsilon_s)\epsilon_s \bar{l}$ and K, solve (w, r) by $w = (1-\alpha)\left(\frac{K}{N}\right)^{\alpha}, \quad r = \alpha\left(\frac{K}{N}\right)^{\alpha-1} - \delta$
- Step 2: Given (r, w), solve a DP problem to obtain policy function $g(k, \epsilon)$.
- Step 3: Given policy function $g(k, \epsilon)$ and transition matrix P, solve the stationary distribution $\lambda(k, \epsilon)$.
- Step 4: From $\lambda(k, \epsilon)$ and $g(k_n, \epsilon_s)$, calculate capital supply

$$K^S = \sum_{s=1}^{S} \sum_{n=1}^{N} \lambda(k_n, \epsilon_s) g(k_n, \epsilon_s)$$

- Step 5: Define $f(K) = K - K^S$, which can be interpreted as excess demand for capital

Hence, by market clear condition, excess demand is zero: $f(K) = 0$.

Solve for Equilibrium

Again, we have an equation solving problem. Apply equation solving methods to solve the equilibrium.
For example: A Dampened Fixed Point Iteration.
Procedure:

- Step 0: Choose an initial conjecture for capital demand $K^0 > 0$, a stopping criterion $\varepsilon > 0$, and a parameter $\gamma \in (0, 1]$.
- Step 1. In Iteration $0 \leq j \leq J$, start with K^j and compute r^j and w^j from pricing functions.
- Step 2. Given $\left(r^j, w^j\right)$, compute the household problem to get $g^j(k, s)$ and associated stationary distribution $\lambda^j(k, s)$.
- Step 3. Calculate capital supply $\widehat{K}^j = \sum_{k,s} \lambda^j(k, s)k$
- Step 4. If $\left|K^j - \widehat{K}^j\right| \leq \varepsilon$, stop. Otherwise, let $K^{j+1} = (1 - \gamma)K^j + \gamma\widehat{K}^j$ and go back to step 1