READING NOTE ON MACRO-FINANCE MODELS

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Contents

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1. Introduction

This section introduces fundamental problems in finance and the linkage between macro-finance. Essentially, finance is transferring resources across time/state (not necessarily in a fair/efficient way though). One fundamental problem studied in finance is the separation between resources and talent/technology. Surprisingly, macroeconomics is also a subject devoted to intertemporal transfer of resources, i.e. Euler equation for household reflects trade-off between consumption (utilizing resource today) and saving (utilizing resource tomorrow); Euler equation for firm reflects trade-off between dividend pay-out (today) and investment (tomorrow). The finance problems are sensitive to macroeconomic conditions, and (failure in) solution to these problem in turn affects macroeconomic condition.

The following case is a good example.

Set-up

An entrepreneur has a brilliant business idea. To implement the idea, fund of 2 dollar is needed. Unfortunately, the entrepreneur has no saving, so she needs external fund (equity or debt) to finance the project. The business project is risky, such that return to investment is

$$R = \begin{cases} 10, & \text{if succeed} \\ 4, & \text{if fail} \end{cases} \tag{1}$$

Option 1: Equity Finance. Suppose outsider investors claim 50% of shares. This means that no matter the results of the project, the entrepreneur can only claim 50% of the return, which is

$$R_e^{equity} = R_c^{equity} = \begin{cases} 5, & \text{if succeed} \\ 2, & \text{if fail} \end{cases} \tag{2}$$

Option 2: Debt Finance. Suppose risk free interest rate is zero. This means that the entrepreneur only need to pay back 2 and keep

$$R_e^{debt} = \begin{cases} 8, & \text{if succeed} \\ 2, & \text{if fail} \end{cases} \tag{3}$$

Things to consider

- valuation and bargaining power: i.e. split of the pie.
  - What if there is only one creditor who can make a take-it-or-leave-it offer?
  - What if there are multiple creditor competing to be the outside investor?
  - What if the creditor is risk neutral?
  - What if the creditor is risk averse?
- information disclosure: i.e. incentive to lie.
  - Suppose the state of the project (succeed or fail) is only observable by the entrepreneur, what's his incentive to lie?
  - Does it vary with method of external finance (equity or debt)?
- contract enforcement: i.e. cost to honor the contract.
  - Suppose when the entrepreneur refuses to honor the contract, the creditor can claim the whole company after paying a cost. How does this cost affect the analysis?

Macro-Finance Linkage

Suppose when the economy is in recession, return to the business project becomes:

$$r = \begin{cases} 7, & \text{if succeed w.p. } 1/2 \\ 1, & \text{if fail w.p. } 1/2 \end{cases} \tag{4}$$

Also suppose existence of competitive and risk-neutral outside investors. This implies:
Option 1: Equity Finance in Recession. Suppose outsider investors claim 50% of shares. This means that no matter the results of the project, the entrepreneur can only claim 50% of the return, which is

$$r_e^{equity} = r_c^{equity} = \begin{cases} 3.5, & \text{if succeed w.p. } 1/2 \\ 0.5, & \text{if fail w.p. } 1/2 \end{cases} \tag{5}$$

Option 2: Debt Finance in Recession. Suppose risk free interest rate is zero. This means that the entrepreneurs only have positive return (=4) when the project succeed, while go bankrupt and have to transfer the company to outside creditor when the project fail.

$$r_c^{debt} = \begin{cases} 3, & \text{if succeed w.p. } 1/2 \\ 1, & \text{if fail w.p. } 1/2 \end{cases} \tag{6}$$

Now let's revisit some of the factors considered before:

- valuation and bargaining power:
  - What if the creditor is risk averse?
- information disclosure:
  - Suppose the state of the project (success or failure) is only observable by the entrepreneur, what's his incentive to lie?
  - Does the result vary with method of external finance (equity or debt)?
  - Suppose if the creditor can pay 1 dollar to verify the true outcome of investment, how does this cost affect the analysis?

Quick Messages

- Equity is more information-sensitive than debt and thus demands more disclosures.
- M-M theory often breaks down with market friction or incompleteness.
- (Macro-finance linkage) macroeconomic states often have an impact on investment and financing choice, which often in turn affects aggregate economy, i.e. positive feedback.

2. Information Asymmetry I: Moral Hazard Problem

2.1. *** Innes (1990, JET).** This paper shows the optimal incentive efficiency of *debt financing* under *moral hazard* and *limited liability.*

2.1.1. *Settings.* Timeline: two-date (static) model

- date 0:
  - Entrepreneur has no fund. Investor provides investment fund I;
  - Entrepreneur puts effort a to the project
- date 1:
  - The project produces a stochastic payoff q with p.d.f dependent on effort: $f(q|a)$. The payoff distribution $f(q)$ has monotone likelihood ratio property ($MLRP$), which means that with higher payoff, the distribution of payoff will be more sensitive to effort.
  $$\frac{\partial}{\partial q}(\frac{f_a(q|a)}{f(q|a)}) > 0 \tag{7}$$
  - Investor gets a repayment contingent on payoff $r(q)$.

Investor's payoff function r(q) has two properties:

- 1) 0 ≤ r(q) ≤ q.
  - r(q) cannot be less than 0 because of limited liability.
  - r(q) cannot be greater than q because the investor cannot require more payoff than the profits available.
- r'(q) ≥ 0.
  - This requires more payoff to the investor when the project payoff is higher. Because if this is not the case, then the entrepreneur can always borrow money to pretend a higher profit and give investor less payoff.

Risk-neutral entrepreneur's utility function is given by $v(w, a) = w - \varphi(a)$ where w is income and a is effort, and $\varphi', \varphi'' > 0$.

2.1.2. *Entrepreneur's problem.*

$$\max \int_0^{\overline{q}} (q - r(q))f(q|a)dq - \varphi(a)$$
$$\begin{aligned} s.t. \int_0^{\overline{q}} r(q)f(q|a)dq &\geq I \qquad \text{(IR)} \\ 0 \leq r(q) &\leq q. \\ r'(q) &\geq 0 \end{aligned} \tag{8}$$

- **First-Best**
  If the entrepreneur can finance internally, the problem becomes
  $$\begin{aligned} &\max \int_0^{\overline{q}} qf(q|a)dq - \varphi(a) - I \\ = &\max \int_0^{\overline{q}} (q - I)f(q|a)dq - \varphi(a) \end{aligned} \tag{9}$$

since $\int_0^{\overline{q}} f(q|a)dq = 1$.

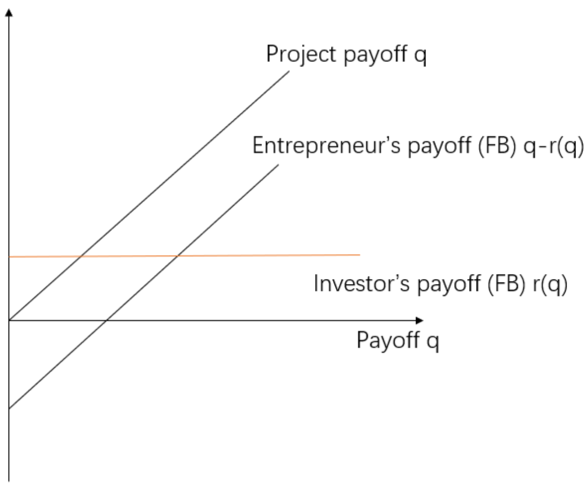


FIGURE 1. First-best Contract

To achieve the first-best, the entrepreneur would like to give the investor a payoff $r(q) = I$, a constant repayment or as flat as possible.
But with the requirement $r(q) \leq q$, the first-best cannot be achieved.

- To satisfy r(q) $\leq$ q and give the investor a payoff as flat as possible, we come up with a payoff function.

$$r(q) = \begin{cases} 0 & \text{if } q > z \\ q & \text{if } q \leq z \end{cases} \tag{10}$$

  - The payoff function satisfies the monotone likelihood ratio property ($MLRP$). It promises higher
  - The payoff function doesn't satisfy $r'(q) \geq 0$ for investor.

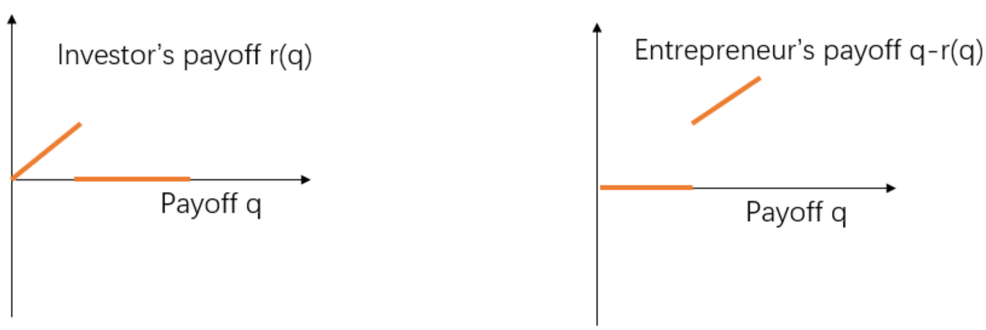


FIGURE 2. Infeasible Contracts

- To make $r'(q) \geq 0$, we now have

$$r(q) = \begin{cases} D & \text{if } q > D \\ q & \text{if } q \leq D \end{cases} \tag{11}$$

where D is given by the IR constraint of investor.

$$\int_0^D qf(q|a)dq + [1 - F(D|a)]D = I \tag{12}$$

and a is given by IC constraint of entrepreneur.

$$\int_D^q (q - D)f_a(q|a)dq = \varphi'(a) \tag{13}$$

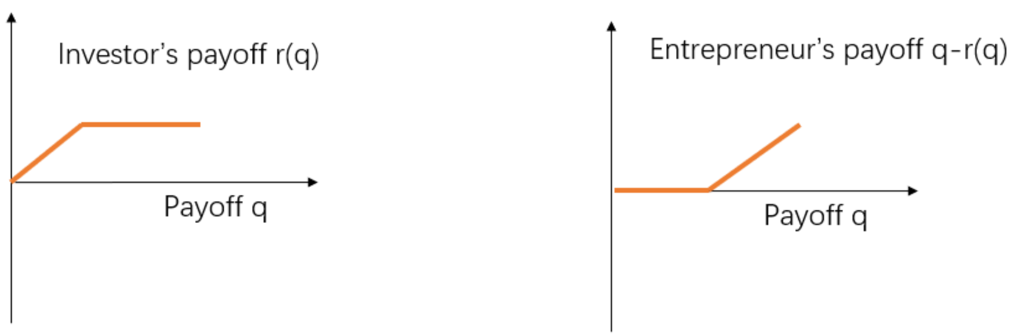


FIGURE 3. Optimal Contracts

2.1.3. *Summary.*

- Debt is optimal in terms of providing incentive.
- Debt is sub-optimal in terms of effort provision relative to *first-best*:
  - FOC of entrepreneur's problem:

$$\int_0^{\overline{q}} (q - r(q))f_a(q|a)dq = \varphi'(a) \tag{14}$$

  - FOC of first-best:

$$\int_0^{\overline{q}} (q - I)f_a(q|a)dq = \varphi'(a) \tag{15}$$

  Since $\int_0^{\overline{q}} r(q)f(q|a)dq \geq I$, effort a in debt contract is smaller than effort in first best.
- A debt contract provides the best incentives for effort provision by extracting as much as possible from the entrepreneur under low performance and by giving her the full marginal return from effort provision in high-performance states where revenues are above the face value of the debt.

2.2. *** Rajan (1992, JF).** This paper studies the benefit and cost of bank loans relative to arm-length debt under *moral hazard*.

2.2.1. *Settings.* Timeline: three-date (static) model

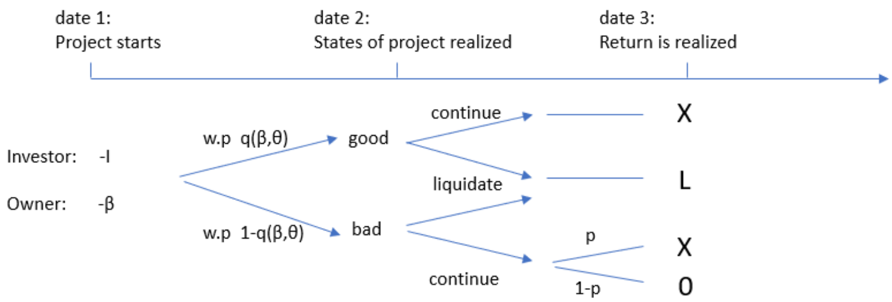


FIGURE 4. Timeline

- date 0:
  - Entrepreneur raise investment fund from external debt I;
  - Entrepreneur exerts effort β;
- date 1:
  - State is realized
  - Entrepreneur learns whether it is good state or bad state
  - Entrepreneur decides whether to continue the project or liquidate and get L
- date 2:
  - Good state: project produces payoff X with probability 1
  - Bad state: project produces payoff X with probability of p_B and 0 otherwise
  - Entrepreneur decides whether to continue the project or liquidate and get L

We assume that $p_B X < L \leq I < X$. Probability $q(\beta, \theta)$ is a function of effort β and exogenous determinants θ. $q_1(\beta, \theta) > 0$ and $q_{11}(\beta, \theta) < 0$.

2.2.2. *First-Best.* The entrepreneur will continue in good state and liquidate in bad state.

$$\max_{\beta} q(\beta)X + (1 - q(\beta))L - I - \beta \tag{16}$$

F.O.C: $q'(\beta) = \frac{1}{X-L}$. The payoff gap of good state and bad state is $X - L$.

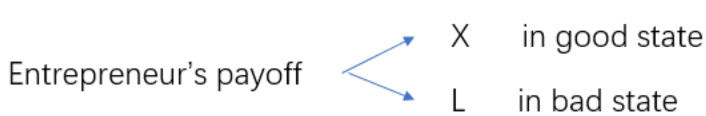


FIGURE 5. First-Best Contract

We will consider three types of external finance:

- Bonds ("arm-length debt")
- Bank loan – short-term
- Bank loan – long-term

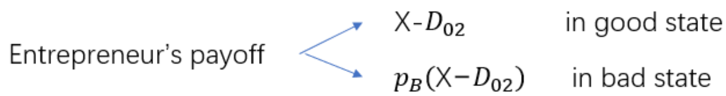


FIGURE 6. Bond Financing

2.2.3. *Bond Financing.* Bondholders cannot observe the state. When it is good state, the entrepreneur will continue the project. When it is bad state, the entrepreneur will get 0 payoff if liquidate ($L \leq I$) and get $p_B(X - D_{02})$ if continue. So they will choose to continue no matter what the state is.

$$\begin{aligned} &\max_{\beta} q(\beta)(X - D_{02}) + (1 - q(\beta))p_B(X - D_{02}) - \beta \\ s.t. \quad & q(\beta)D_{02} + (1 - q(\beta))p_B D_{02} = I(\text{IR Condition}) \end{aligned} \tag{17}$$

F.O.C: $q'(\beta) = \frac{1}{(1-p_B)(X-D_{02})}$. The payoff gap of good state and bad state is $(1-p_B)(X-D_{02})$. Here $D_{02} > I > L$, so $(1 - p_B)(X - D_{02}) < X - L$. The gap is smaller. Since $q" < 0$, the effort β will decrease comparing to first best.
This is because the owner continues in the bad states, forcing the rational lender to demand a higher face value than if the continuation decision were efficient. This reduces the surplus available to the owner in the good state.
(Feedback Loop) When the effort decrease, $q(\beta)$ will decrease. To satisfy the IR condition, the investor will increase the face value of debt D_{02}. With larger D_{02} in F.O.C, the effort will decrease again. This will be a feedback loop with less effort and more face value of debt.

2.2.4. *Short-term Bank Loan.* Bank can observe the state and renegotiate the contract after knowing the state. The bank will ask the firm to liquidate in bad state. In good state, the bank can hold up the owner and demand a share of the surplus in return for the loan to continue the project.
(Bargaining Game) In good state, the entrepreneur gets $\mu(X - L)$ while the bank gets $X - \mu(X - L) = L + (1 - \mu)(X - L)$. As long as $\mu(X - L) > 0$, both the owner and the bank are better off.

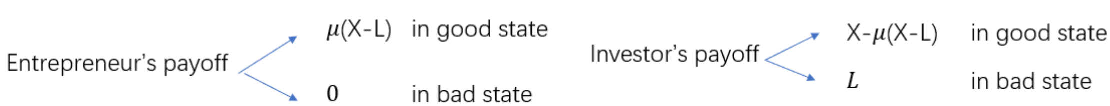


FIGURE 7. Bank Financing: Short Term

$$\max_{\beta} q(\beta)\mu(X - L) + (1 - q(\beta))0 - \beta \tag{18}$$

F.O.C: $q'(\beta) = \frac{1}{\mu(X-L)}$. The payoff gap of good state and bad state is $\mu(X - L)$ which is less than the first best. So the effort β is less than the first best. The gap shrinks because some of the surplus are taken by investor in good state and this distort the firm's incentive for effort.

2.2.5. *Long-term Bank Loan.* Bank can observe the state. But the loan is long-term now. In good state, the firm will continue the project. In bad state, the firm can hold up the bank and demand a share of the surplus$(L - p_B X)$ in return to liquidate the project.
(Bargaining Game) In bad state, the entrepreneur gets $p_B(X - D_{0B}) + \mu(L - p_B X)$ while the bank gets $p_B D_{0B} + (1 - \mu)(L - p_B X)$. As long as $\mu(L - p_B) > 0$, both the owner and the bank are better off by liquidation.

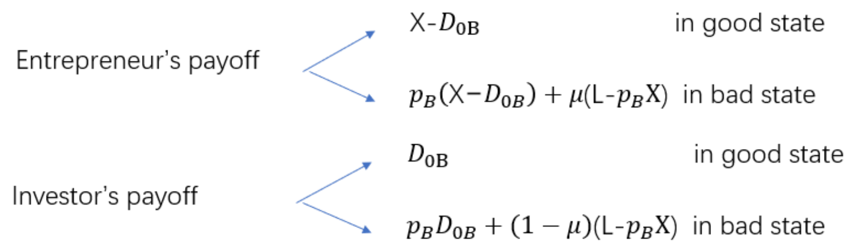


Figure 8. Bank Financing: Long Term

$$\begin{aligned} &\max_{\beta} q(\beta)(X - D_{0B}) + (1 - q(\beta))[p_B(X - D_{0B}) + \mu(L - p_B X)] - \beta \\ s.t. \quad & q(\beta)D_{0B} + (1 - q(\beta))[p_B D_{0B} + (1 - \mu)(L - p_B X)] = I(\text{IR condition}) \end{aligned} \tag{19}$$

F.O.C: $q'(\beta) = \frac{1}{(1-p_B)(X-D_{0B})-\mu(L-p_B X)}$. The payoff gap of good state and bad state is $(1 - p_B)(X - D_{0B}) - \mu(L - p_B X)$. Compare with the payoff gap of bond $(1 - p_B)(X - D_{02})$, the payoff gap of long-term bank loan is smaller because firm is better off in bad state.
For investor, compare the IR condition and we can find that investor is better off in bad state. So the face value of debt D_{0B} is lower than the face value of bond D_{02}
(Trade-off) After renegotiation, the liquidation problem is solved and both the owner and the investor are better off (*ex post*). But the bargaining power makes the firm's incentive decrease (*ex ante*).

2.2.6. *Comparison.* 1. Short-term bank loan is worse than the first best because bank will ask for a share of the surplus in good state and it will hurt the incentive of entrepreneur.
2. Long-term bank loan is worse than bond because when entrepreneur has the bargaining power in bad state they will make less effort.
3. Bank loans are better than bond in the sense that they solve the liquidation problem but at a cost of distortion to effort incentives
In summary, which one is better depends on the parameter.

2.2.7. *Extension.* When the firm use a mix of internal finance and external finance, it could probably solve the liquidation problem but not the incentive problem.

2.3. *** Holmstrom and Tirole (1997, QJE).** This paper studies a moral hazard problem in a *general equilibrium* model. The paper also introduces a role of financial intermediate.

2.3.1. *Settings.* Timeline: two-date (static) model

- date 0: An investment project costs a fixed I units of capital.
  - Entrepreneur owns A units of capital, where $A \leq I$.
  - Entrepreneur raises fund from bank or investors to finance $I - A$ units of capital.
- date 1: Investment returns are realized.
  - R for success with probability p
  - 0 for failure with probability 1-p

We have three types of agents:

- Firm: There is a continuum of firms with different amounts of capital A (heterogeneity), with c.d.f $G(A)$.
  - Firms are run by entrepreneurs, who, in the absence of proper incentives or outside monitoring, may deliberately reduce the probability of success in order to enjoy a private benefit B or b with $B > b$.
  - When entrepreneur shirks, the probability of success will decrease from P_H to P_L.
- Investor: uninformed investors demand an expected rate of return γ.
- Bank: Bank demands an expected rate of return β
  - $\beta > \gamma$: Bank has extra cost in monitoring.
  - Bank can monitor the firm with a cost of c.
  - When firm is monitored by bank, it can only get private benefit b if they shirk.

Project	the Good	the Bad	the Ugly
Private Benefit	0	b	B
Prob. of Success	p_H	p_L	p_L

TABLE 1. Moral Hazard Problem

We assume that only good project can get positive NPV.

$$p_H R - \gamma I > 0 > p_L R - \gamma I + B > p_L R - \gamma I + b \tag{20}$$

2.3.2. *Scenario I: Direct Finance.* When the firm finances a project only from uninformed investor (direct finance), total return upon success is divided into two parts: $R = R_f + R_u$, where R_f is for the firm and R_u is for the investor. The firm contributes A and the investor contributes $I - A$ units of capital at the beginning of the project.

Firm: A necessary condition for direct finance is that the firm prefers to be diligent:

$$p_H R_f \geq p_L R_f + B \qquad \text{(IC)} \tag{21}$$

Then firm needs payoff $R_f \geq \frac{B}{\Delta p}$ to make sure it work hard.

Investor: The return left for investor is at most $R_u \leq R - \frac{B}{\Delta p}$.
We also need to make sure that investor want to participate.

$$
\begin{aligned}
& (I-A)\gamma \leq p_H(R - \frac{B}{\Delta p}) \qquad \text{(IR)} \\
\Leftrightarrow \quad & (I-A) \leq \frac{p_H}{\gamma}(R - \frac{B}{\Delta p}) \\
\Leftrightarrow \quad & A \geq \overline{A}(\gamma) = I - \frac{p_H}{\gamma}(R - \frac{B}{\Delta p})
\end{aligned} \tag{22}
$$

This implies that to make sure the firm works hard, the firm must have **high enough shares**, so that the moral hazard problem disappears.

2.3.3. *Scenario II: Direct + Indirect Finance.* When the firm finances a project from both uninformed investors (direct finance) and the bank (indirect Finance), total return upon success is divided into three parts: $R = R_f + R_u + R_m$, where R_f is for the firm, R_u is for the investor and R_m is for the bank. The firm contributes A, the investor contributes $I - A - I_m$ and the bank contributes I_m units of capital at the beginning of the project.
Firm: A necessary condition for indirect finance is that the firm prefers to be diligent (assuming that bank will monitor):

$$p_H R_f \geq p_L R_f + b \qquad \text{(IC)} \tag{23}$$

Then firm needs payoff $R_f \geq \frac{b}{\Delta p}$ to make sure it work hard.
Bank: Bank will get a private benefit of c if they do not monitor the firm. To make sure the bank will monitor, it requires:

$$p_H R_m \geq p_L R_m + c \qquad \text{(IC)} \tag{24}$$

Then bank needs payoff $R_m \geq \frac{c}{\Delta p}$ to make sure it will monitor. The rate of return on intermediary capital is $\beta = p_H R_m / I_m$. Bank breaks even.

$$I_m = \frac{p_H}{\beta} R_m = \frac{p_H}{\beta}\frac{c}{\Delta p} \tag{25}$$

Investor: The return left for investor is at most $R_u \leq R - \frac{b+c}{\Delta p}$.
We also need to make sure that investor wants to participate.

$$
\begin{aligned}
(I - I_m - A)\gamma &\leq p_H(R - \frac{b+c}{\Delta p}) \qquad \text{(IR)} \\
I - \frac{p_H}{\beta}\frac{c}{\Delta p} - A &\leq \frac{p_H}{\gamma}(R - \frac{b+c}{\Delta p}) \\
A \geq \underline{A}(\gamma, \beta) &= I - \frac{p_H}{\beta}\frac{c}{\Delta p} - \frac{p_H}{\gamma}(R - \frac{b+c}{\Delta p})
\end{aligned} \tag{26}
$$

In summary, firms with abundant net worth ($A > \bar{A}(\gamma)$) finance their investment using direct finance only, while firms with intermediate net worth, i.e. $A \in [\underline{A}(\gamma, \beta), \bar{A}(\gamma)]$, finance their investment using both direct and indirect finance. The firms with lowest net worth cannot be financed.

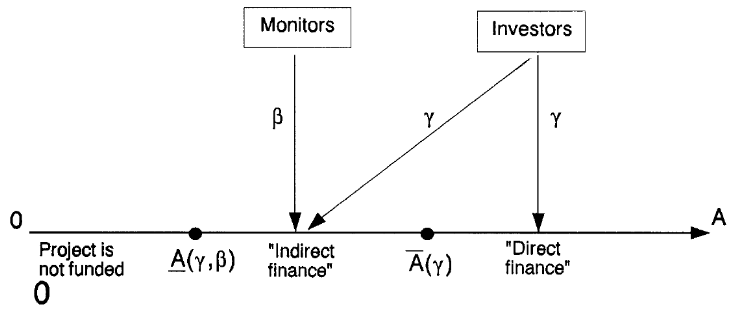


FIGURE 9. Credit Market Equilibrium

Equilibrium: In equilibrium, we solve $\{\gamma, \beta\}$ from credit market clearing conditions.
The aggregate demand for informed capital (bank) is equal to the aggregate supply (K_m):

$$\int_{\underline{A}}^{\overline{A}} I_m(\beta) dG(A) = \int_{\underline{A}}^{\overline{A}} \frac{p_H}{\beta}\frac{c}{\Delta p} dG(A) = K_m \tag{27}$$

The aggregate demand for uninformed investor's capital is equal to aggregate saving ($S(\gamma)$):

$$\int_{\overline{A}}^{\infty} (I - A) dG(A) + \int_{\underline{A}}^{\overline{A}} (I - A - \frac{p_H}{\beta}\frac{c}{\Delta p}) dG(A) = S(\gamma) \tag{28}$$

where $\underline{A}$ and $\overline{A}$ are functions of (β, γ), and saving function $S(\gamma)$ increases with interest rate (γ). Using the above two equations we can solve for (β, γ). The prices of capitals are endogenous in general equilibrium model.
Combine the above two equations we can get the aggregate demand for external finance is equal to the aggregate supply of external finance.

$$\int_{\underline{A}}^{\infty} (I - A) dG(A) = K_m + S(\gamma) \tag{29}$$

Lastly, the aggregate investment in the economy is:

$$K \equiv I \int_{\underline{A}(\gamma,\beta)}^{\infty} dG(A) = \int_{0}^{\infty} AG(A) + K_m + s(\gamma) \tag{30}$$

where $S(\gamma) \equiv \int_0^{\underline{A}} AG(A) + s(\gamma)$.
Comparative Static: We analysis the effect of change in K_m on equilibrium objects.
Intermediate Capital (K_m): Given γ unchanged, if there is a reduction in K_m, from indirect credit market clearing condition (27) we know β must increase . Since β increases, as $\underline{A}$ is an increasing function of β, total investment declines. This suggests an important role played by financial intermediate.
With endogenous interest rate γ demanded by uninformed investors, we can show from equation (30) that $\underline{A}$ must increase, i.e. aggregate investment must decrease.

- Suppose $\underline{A}$ decreases, so that LHS of equation (30) increases. then $s(\gamma)$ must increase. This implies γ increases, as saving function $s(\gamma)$ increases with γ. Recall $\underline{A}(\beta, \gamma)$ increases in both γ and β, for $\underline{A}$ to decrease we must have decreasing β. From indirect credit market clearing condition (27), if β and $\underline{A}$ decrease, the LHS must increase, thus the RHS, K_m must increase as well. This contradicts with reduction in K_m.

2.4. *** Farhi and Tirole (2012, AER).** This paper studies a *collective moral hazard* problem in general equilibrium. The paper shows that *ex post* moral hazard problem of the government to bailout an individual bank, when the entire banking system is in trouble, induces *ex ante* moral hazard problem of each bank that the exposure to aggregate risk increases when it expects other banks to do the same.

2.4.1. *Settings.* agents: bankers, consumers and a central bank

- banker:
  - risk-neutral: U = $c_0 + c_1 + c_2$
  - linear technology
- consumer (investor):
  - born at t = 0 or 1;
  - consume at t+1: $u_t = c_{t+1}$
  - deep pocket: $s >> 0$
- central bank (planner)
  - maximize weighted welfare of bankers and consumers;
  - control interest rate

timeline: three-date model

- date 0:
  - receive endowment A
  - borrow in short-term debt (i-A at interest rate 1)
  - invest: -i
- date 1:
  - safe cash flow πi
  - aggregate uncertainty: intact (α) or distress ($1-\alpha$)
  - intact: + $\rho_1 i$

$$\rho_1 i = \underbrace{\rho_0 i}_{\text{pledgeable}} + \underbrace{(\rho_1 - \rho_0) i}_{\text{agency cost}}$$

  - distress: -j (reinvestment at interest rate R)
- date 2:
  - distress: + $\rho_1 j$

$$\rho_1 j = \underbrace{\rho_0 j}_{\text{pledgeable}} + \underbrace{(\rho_1 - \rho_0) j}_{\text{agency cost}}$$

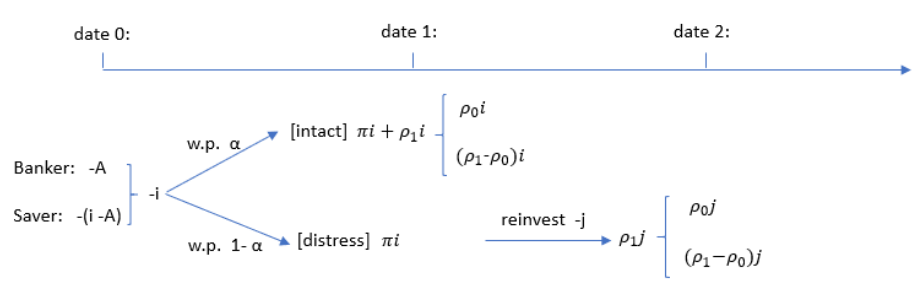


FIGURE 10. Timeline

Assumption 0 & 1

- project can only be downsized in crisis:

$$j \leq i \tag{31}$$

- incentive to invest at date 0:

$$\pi + \rho_1 > 1 + (1-\alpha) \tag{32}$$

2.4.2. *Banker's Problem.* The bank issues state-contingent short-term debt, whose repayment is

- no crisis: $\pi i + \rho_0 i$.
- at crisis: di (where $d \leq \pi$) $\Rightarrow$ liquidity holding: $xi \equiv (\pi - d)i$
- reinvestment constraint:

$$j \leq \underbrace{(\pi - d)i}_{self-financed} + \underbrace{\frac{\rho_0 j}{R}}_{borrowed} \quad \& \quad j \leq i$$

$$\Rightarrow \quad j = \min\{\frac{xi}{1-\frac{\rho_0}{R}}, i\}$$

  - lower interest rates ($R\downarrow$) facilitate refinancing ($j\uparrow$)
  - we focus on the case: $\frac{x}{1-\frac{\rho_0}{R}} \leq 1$
  - we assume $d \geq \pi - (1-\frac{\rho_0}{R})$, or liquidity ratio: $x \in [0, 1-\frac{\rho_0}{R}]$.

$$j = \frac{xi}{1-\frac{\rho_0}{R}} \tag{33}$$

- IR constraint over outside investors

$$i - A = \alpha(\pi i + \rho_0 i) + (1-\alpha)di$$
$$\Rightarrow \quad i = \frac{A}{1 - \pi - \alpha r_0 + (1-\alpha)x} \tag{34}$$

- banker chooses x (or d) between $[0, 1-\frac{\rho_0}{R}]$ to maximize:

$$[\alpha(\rho_1 - \rho_0)i + (1-\alpha)(\rho_1-\rho_0)j] \tag{35}$$

  subject to i and j given in equation 33 and 34
- Thus bankers optimization problem is:

$$\max_x(\rho_1 - \rho_0)[\frac{\alpha + (1 - \alpha\frac{x}{1-\rho_0/R})}{1 - \pi - \alpha\rho_0(1-\alpha)x}]$$

- optimal borrowing policy at date 0:
  - load up on short-term debt $x = 0$ (or $d = \pi$) iff

$$\alpha + \pi > 1 + \rho_0(1/R - 1) \tag{36}$$

  - takes on just enough short-term debt to be able to continue full scale: $x = 1 - \rho_0/R$ (or $d = \pi - 1 + \rho_0/R$) otherwise.

Assumption 2

- bankers prefer to limit the amount of short-term debt to have enough liquidity to continue at full scale

$$\alpha + \pi \leq 1 \tag{37}$$

2.4.3. *Consumer's Problem.* Saving b/w day-0 and day-1 (trivial)

- interest rate = 1

Saving b/w day-1 and day-2

- natural rate of interest = 1
- storage rate: 1
- tax on storage rate : 1-R (lump-sum rebate at date 2)
- interpretation: policy interventions that reduce borrowing costs for banks
- $s >> 0 \Rightarrow$ storage > 0

Assumption 3 & 4

- (interest rate distortion) The set of feasible interest rates (R) is $[\rho_0, 1]$. Furthermore, there exists a fixed distortion or dead-weight loss L(R) ≥ 0 when the interest rate R diverges from its natural rate: $L(1) = L'(1) = 0$ and L(R) is decreasing over $[\rho_0, 1]$.
  - simplifies the exposition but plays no substantial role in analysis
  - normalize optimal interest rate under commitment to R = 1
  - lower bound R = ρ_0 protected with pleageable income
  - lowering the interest rate below ρ_0 would only increase the distortion associated with interest policy at no gain
- (consumer welfare) Suppose that date-0 investment is equal to i, and that banks hoard liquidity x and so can salvage $j = xi/(1 - \frac{\rho_0}{R})$ at crisis. Then,
  - if there is crisis at date 1, the date-1 welfare of consumer is

$$V = -(\underbrace{L(R)}_{DWL} + \underbrace{(1-R)\frac{\rho_0 j}{R}}_{subsidy}) \tag{38}$$

  - if there is no crisis at date 1, consumer welfare is $V = -L(1) = 0$.
  - we ignore the welfare of consumer born at day 0: $u = s$.

2.4.4. *Central Bank's Problem.* Interpreting policy on R:

- unconventional monetary policy.
  - extended debt guarantees $\rightarrow$ reduce the rate paid by banks government saves and other borrowers
  - subsidy is paid by taxpayers $\rightarrow$ risk of debt
  - a subsidy from savers to borrowers $\rightarrow$ reduce the marginal borrowing cost of banks
- conventional monetary policy.
  - deposit insurance tends to be under-priced in crisis
  - higher reserves $\rightarrow$ banks lever more through access to cheap retail deposits
  - deposit insurance is backed by taxes on consumers $\rightarrow$ risk of debt
  - a subsidy from taxpayers to borrowing banks $\rightarrow$ reduce the marginal borrowing cost of banks
- conventional monetary policy in NK framework
  - a prolonged reduction of interest rates (aka Japan type)
  - nominal interest rate controlled by CB in NK model + sticky price $\Rightarrow$ real interest rate affected by CB
  - CB can achieve flexible price economy allocation by setting nominal interest rates $\Rightarrow$ real interest rate equals = "natural" interest rate (R=1 in our model)

  - deviation from this rule ⇒ variations in the output gap: $L(R)$ in our model is a reduced form representation

Assumption 5

- objective function of central banker at date 1 (j = i if no crisis) is

$$W = V + \beta j \tag{39}$$

- objective function at date 0 is expected W.
- interpreting βj:
  - stake in continuation to banker
  - the higher j, the better off the banks' borrowers
  - workers in banks and industrial companies better off employed

2.4.5. *Scenario I: Commitment.* We start with a scenario when central bank can commit at date 0 to a specific contingent policy at date 1. Bankers and consumers form expectations regarding the interest rate $R \in [\rho_0, 1]$ that will be set if a crisis occurs. This section analyzes the equilibrium when the date-1 interest rate is chosen at date 0. Ex ante welfare is

$$W^{\text{ex ante}}(R) \equiv \alpha[V(1) + \beta i(R)] + (1-\alpha)[V(R) + \beta j(R)]$$

where

$$V(R) \equiv -\left[L(R) + (1-R)\frac{\rho_0 j(R)}{R}\right]$$

and

$$j(R) = i(R) = \frac{A}{1 - \pi - \alpha\rho_0 + (1-\alpha)\left(1 - \frac{\rho_0}{R}\right)}$$

Using $V(1) = 0$, we can write

$$W^{\text{ex ante}}(R) = \left[\beta - (1-\alpha)\frac{1-R}{R}\rho_0\right] i(R) - (1-\alpha)L(R)$$

Effects of an increase in announced interest rate R at date 0

- reduce distortion in welfare of consumer: $-L(R) \uparrow$
- reduce leverage and investment of banker: $i(R) \downarrow$
- redistribution: banker → consumer

Assumption 6:

- no ex-ante (date-0) wealth transfer condition:

$$\beta \leq 1 - \alpha + 1 - \pi - \rho_0 \tag{40}$$

- ⇒ the following component of $W^{ex\ ante}$ is non-decreasing in R:

$$\left[\beta - (1-\alpha)\frac{1-R}{R}\rho_0\right] i(R) \tag{41}$$

- intuition: social value of 1-unit dollar transfer from consumer to banker is negative

$$-1 + \beta \underbrace{\frac{i(R=1)}{A}}_{leverage} = 1 - \frac{\beta}{1 - \alpha + 1 - \pi - \rho_0} \leq 0 \tag{42}$$

Proposition 1. The optimal interest rate policy under commitment features

$$R^c = 1 \tag{43}$$

2.4.6. *Scenario II: No-Commitment Solution.* Then we consider a scenario when central bank set the interest rate at date 1 with no regard for previous commitments. At date 0, bankers and consumers still form expectations regarding the interest rate $R^* \in [\rho_0, 1]$ that will be set if a crisis occurs. And based on this expectation, the representative bank invests at scale $i(R^*)$, and hoards just enough liquidity $x^* i(R^*)$ to be able to reinvest at full scale in the event of a crisis, where $x^* = 1 - (\rho_0/R^*)$.
No crisis. If there is no crisis, it is optimal to set $R = 1$.
In Crisis. The central bank faces the following trade-off:

- $R \downarrow \Rightarrow j \uparrow$
- $R \downarrow \Rightarrow L(R) \uparrow$
- $R \downarrow \Rightarrow$ redistribution loss: consumer $\rightarrow$ banker
- NB: $R \downarrow$ does not increase continuation scale

As a result,

- the central bank would never set $R < R^*$.
- the central bank has incentive to set $R > R^* \Rightarrow$ cost: forced downsize $j < i$:
$$j = \frac{x^*}{1 - \frac{\rho_0}{R}} i\left(R^*\right) \Leftrightarrow j = \frac{1 - \frac{\rho_0}{R^*}}{1 - \frac{\rho_0}{R}} i\left(R^*\right) \tag{44}$$
- ex post (date-1) welfare $W^{\text{ex post}}\left(R;\ R^*\right)$ in case of crisis:
$$W^{\text{ex post}}\ \left(R; R^*\right) = \underbrace{-L(R)}_{\uparrow \text{ w.r.t. R}} + \underbrace{\left[\beta - (1-R)\frac{\rho_0}{R}\right] \frac{1 - \frac{\rho_0}{R^*}}{1 - \frac{\rho_0}{R}} i\left(R^*\right)}_{\downarrow \text{ w.r.t. R if w} > 0; \quad \uparrow \text{ w.r.t. R if w} \leq 0} \tag{45}$$
where $w \equiv \beta - (1 - \rho_0)$.
- $\mathcal{R}\left(R^*\right)$: the set correspondence defined by
$$\mathcal{R}\left(R^*\right) \equiv \arg\max_R W^{\text{ex post}}\ \left(R; R^*\right) \tag{46}$$
or equivalently, The interest rate R^{nc} is an equilibrium iff the cost exceeds the gain for all interest rates $R > R^{nc}$.

$$\underbrace{\frac{w\rho_0}{1 - \frac{\rho_0}{R}} \left(\frac{1}{R^{nc}} - \frac{1}{R}\right) i\left(R^{nc}\right)}_{\text{cost of R} \uparrow \text{ in terms of} \downarrow \text{ in j}} \geq \underbrace{L\left(R^{nc}\right) - L(R)}_{\text{gain of R} \uparrow \text{ in terms of} \downarrow \text{ in distortion}} \qquad \text{for all } R \in [R^{nc}, 1] \tag{47}$$

Assumption 7

- (Ex post bailout temptation) $w > 0$.
  - It's more tempting to transfer wealth towards bankers ex post than ex ante.
  - if $w \leq 0$, $\mathcal{R}(R^{nc}) = 1$: there is no commitment problem.
  - if $w > 0$, $W^{\text{ex post}}$ can be non-monotone function of R.

Equilibria. The equilibrium set R^{nc} corresponds to the set of fixed points of $R^{nc} \in \mathcal{R}(R^{nc})$
Proposition 2. Every solution R^{nc} corresponds to an equilibrium where investors and banking entrepreneurs correctly anticipate that the central bank will set $R = R^{nc}$ if a crisis occurs, invest at scale $i(R^{nc})$, and issue short-term debt $(\pi - 1 + \rho_0/R^{nc})i(R^{nc})$. Moreover, there exists $\xi > 0$ such that $[1 - \xi, 1] \in \{R^{nc}\}$.

- $R^{nc} = 1 = R^c$ is always an equilibrium of the no-commitment economy.
- There are always other equilibria with $R^{nc} > \rho_0$, which follows directly from $L'(R) = 0$.

- Intuition: the right-hand side of last equation is small compared to the left-hand side for R^{nc} close enough to 1.
- $R^{nc} = \rho_0$ is an equilibrium if

$$-L(\rho_0) - (1-\rho_0)i(\rho_0) + \beta i(\rho_0) \geq 0$$

or equivalently

$$\frac{wA}{1-\pi-\alpha\rho_0} \geq L(\rho_0) \tag{48}$$

Corollary 1. Suppose that last equation holds. Then $R^{nc} = 1$ and $R^{nc} = \rho_0$ are equilibria of the no-commitment economy.

- If agents expect the central bank to adopt a tough stance by setting $R = 1$ in case of crisis, then banks choose a small scale i (1) and hoard enough liquidity $(1-\rho_0)i(1)$ to withstand the shock even if the central bank sets $R = 1$. In turn, the central bank has no incentive to lower the interest rate below 1.
- if agents expect the central bank to adopt a soft stance by setting $R^{nc} = \rho_0$ in case of a crisis, then banks choose a large scale $i(\rho_0)$ and hoard no liquidity. Then, if a crisis occurs, banks can continue at a positive scale only if the central bank sets the interest rate at its lowest possible level $R^{nc} = \rho_0$ and engineers an extreme bailout. In turn, this extreme bailout is the optimal course of action for the central bank.

Strategic Complementarities. Banks' leverage decisions are strategic complements: Each bank's leverage decision has an effect on the other banks through the policy reaction function in case of a crisis.

- The intuition is as follows: Suppose there are two banks in the system. If the other bank chooses a higher leverage $\Rightarrow$ the central bank is more likely to choose lower interest rate (bail-out) in crisis $\Rightarrow$ better for me to choose higher leverage
- *Corollary 2.* Strategic complementarities associated with bigger (higher A), more powerful (higher β), and more strategic banks are stronger.
- *Corollary 3.* Strategic complementarities associated with higher severity of the crisis are stronger.

Endogenous Correlation. So far we have assumed that correlation of distress across all banks is exogenous (and = 1). We now relax this assumption and allow banks to choose the correlation of their distress risk with other banks' distress risk. As a result, banks want to fail when the largest possible number of other banks are failing and correlate their risks with those of other banks. This is because that bailouts take place in states of the world where a large number of banks are in distress, making it cheaper to refinance in these states. In other word, strategic complementarities are present in correlation choices.

2.4.7. *Discussion: Intertemporal Channel.* In this section we provides two foundations for the hazards of low–interest rate policies as sowing the seeds for the next crisis. In particular, we consider deferred costs, associated with the incentive for new borrowers to lever up and increase maturity mismatch, and with the central bank's loss of reputation. We consider an over-lapping generation model with many generations, G_t, $G_t + 1$ etc,

Leverage and Maturity Mismatch. Bailout policy at $G_t \rightarrow$ lower $R_t \rightarrow G_{t+1}$: take an illiquid position and higher leverage $\rightarrow$ distort generation G_{t+1}'s incentives $\rightarrow$ crisis more likely at $t+1$.

Central Bank's Reputation. This can be modeled by introducing a tough type and a soft type. A bailout then reveals the type of the central bank to be soft, raising the likelihood of future bailouts and pushing banks to take on more risk, hoard less liquidity and lever up, resulting in increased economy-wide maturity mismatch and in turn larger bailouts.

3. Information Asymmetry II: Adverse Selection Problem

3.1. *** Myers and Majluf (1984, JFE).** This paper studies an equity finance problem under *adverse selection* when borrowers have private information on their type. The paper also provides rationale for the *'pecking order theory'*.

3.1.1. *Settings.* Timeline: two-date (static) model

- date 0:
  - Firm could be in good state or bad state with equal probability.
  - Firm has asset in place (γ) and a new investment opportunity.
  - The new investment project cost 0.5.
- date 1: Investment returns are realized.
  - 1 for success w.p. η_G/η_B in good/bad state
  - 0 for failure w.p. $1-\eta_G/1-\eta_B$ in good/bad state

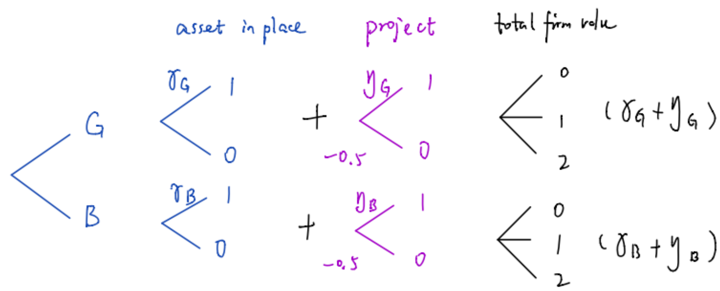


Figure 11. Timeline

We assume that $\eta_G > \eta_B \geq 0.5$, i.e. the new project has positive NPV in both states.

3.1.2. *First-Best.* The project should be implemented in both states as

$$\begin{aligned} NPV_G &= \eta_G - 0.5 > 0 \\ NPV_B &= \eta_B - 0.5 > 0 \end{aligned} \tag{49}$$

3.1.3. *Asymmetric Information: Pooling Equilibrium.* Suppose the state is private information to the firm, in other word, creditors cannot observe the true state. Let's first solve the pooling equilibrium, i.e. the project is undertaken in both good state and bad state.
Equity Financing. Suppose the firm uses equity financing and give investor a proportion of shares. To make sure the investor want to participate, the following IR condition must be satisfied:

$$\begin{aligned} & 0.5 = a[0.5(\gamma_G+\eta_G)+0.5(\gamma_B+\eta_B)] \qquad \text{(IR condition)} \\ \Leftrightarrow \quad & a = \frac{0.5}{0.5(\gamma_G+\eta_G)+0.5(\gamma_B+\eta_B)} \end{aligned} \tag{50}$$

In this case the investor overvalue bad state firm and undervalue good state firm. Bad firm is definitely willing to participate because they get benefit from this pooling equilibrium. For good firm, the IR condition is

$$\begin{aligned} (1-a)(\gamma_G+\eta_G) &> \gamma_G \\ (1-\frac{1}{(\gamma_G+\eta_G)+(\gamma_B+\eta_B)})(\gamma_G+\eta_G) &> \gamma_G \\ (\gamma_G+\eta_G)-\frac{(\gamma_G+\eta_G)}{(\gamma_G+\eta_G)+(\gamma_B+\eta_B)} &> \gamma_G \\ (\gamma_G+\eta_G)-\frac{1}{2}-\frac{1}{2}\frac{(\gamma_G+\eta_G)-(\gamma_B+\eta_B)}{(\gamma_G+\eta_G)+(\gamma_B+\eta_B)} &> \gamma_G \\ \eta_G-0.5 > 0.5\frac{(\gamma_G+\eta_G)-(\gamma_B+\eta_B)}{(\gamma_G+\eta_G)+(\gamma_B+\eta_B)} & \end{aligned} \tag{51}$$

Rewrite this as a function of γ_G

$$\begin{aligned} \eta_G-0.5 &> 0.5\frac{(\gamma_G+\eta_G)-(\gamma_B+\eta_B)}{(\gamma_G+\eta_G)+(\gamma_B+\eta_B)} \\ (\eta_G-0.5)[(\gamma_G+\eta_G)+(\gamma_B+\eta_B)] &> 0.5[(\gamma_G+\eta_G)-(\gamma_B+\eta_B)] \\ (\eta_G-0.5-0.5)(\gamma_G+\eta_G) &> -\eta_G(\gamma_B+\eta_B) \\ \gamma_G+\eta_G &< \frac{\eta_G}{1-\eta_G}(\gamma_B+\eta_B) \qquad \text{since } \eta_G<1 \\ \gamma_G < \overline{\gamma_G} &= \frac{\eta_G}{1-\eta_G}(\gamma_B+\eta_B)-\eta_G \end{aligned} \tag{52}$$

If the good firm has large asset in place, then the good firm will not participate in a pooling equilibrium. This means that a positive-NPV project will not be undertaken.

3.1.4. *Asymmetric Information: Separating Equilibrium.* Suppose the state is private information to the firm, in other word, creditors cannot observe the true state. Let's first solve the pooling equilibrium, i.e. the project is undertaken only in bad state but not in good state [1].

Equity Financing. Suppose the firm uses equity financing and give investor a proportion of shares. Now the investor's belief is that only bad state firm participate. To make sure the investor want to participate

$$\begin{aligned} 0.5 &= a(\gamma_B + \eta_B) \qquad \text{(IR condition)} \\ a &= \frac{0.5}{\gamma_B + \eta_B} \end{aligned} \tag{53}$$

Bad state firm are definitely willing to participate:

$$\begin{aligned} (1-a)(\gamma_B + \eta_B) &= (1 - \frac{0.5}{\gamma_B + \eta_B})(\gamma_B + \eta_B) \\ &= \gamma_B + \eta_B - 0.5 > \gamma_B \end{aligned} \tag{54}$$

To make sure that good state firm will not participate, we must have:

$$\begin{aligned} (1-a)(\gamma_G + \eta_G) &< \gamma_G \\ (1 - \frac{0.5}{\gamma_B + \eta_B})(\gamma_G + \eta_G) &< \gamma_G \\ (1 - \frac{0.5}{\gamma_B + \eta_B})\eta_G &< \gamma_G - (1 - \frac{0.5}{\gamma_B + \eta_B})\gamma_G \\ (\gamma_B + \eta_B - 0.5)\eta_G &< 0.5\gamma_G \\ \gamma_G > \underline{\gamma_G} &= (2\gamma_B + 2\eta_B - 1)\eta_G \end{aligned} \tag{55}$$

In this case, when the firm announces a new equity issue, investors learn not only that it has a new investment project available but also that the firm is in bad state. As a result, firm value drops from $0.5\gamma_G + 0.5(\gamma_B + \eta_B - 0.5)$ to $\gamma_B + \eta_B - 0.5$. Thus in this separating equilibrium there is a negative stock price reaction to the announcement of a new equity issue.

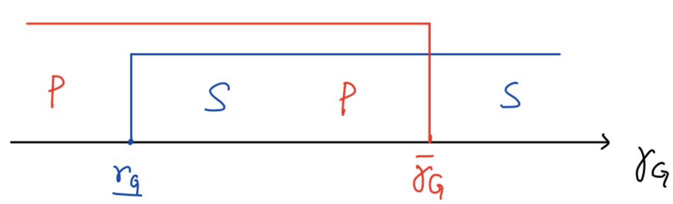


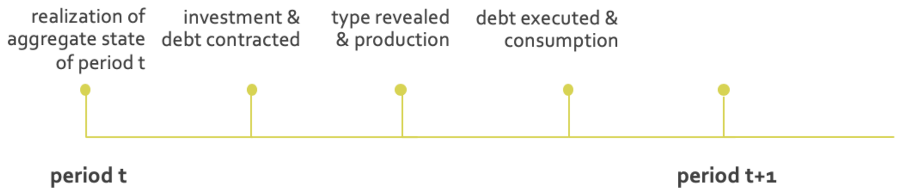
FIGURE 12. Regimes: Separating and Pooling Equilibrium

When $\gamma_G \in [\underline{\gamma_G}, \overline{\gamma_G}]$, both pooling and separating equilibria exist. Here investors' beliefs can be *self-fulfilling*: the firm in good state issues equity if and only if the market thinks that it does. If the market believes that the firm in good state issues equity, it is ready to give the firm more favorable terms, which in turn makes it more attractive for the firm to issue equity.

[1]The opposite is not possible in this context because bad firm will always want to mimic good firm.

3.1.5. *Conclusion.* With information asymmetry, debt finance would be better than equity finance. But debt finance is dominated by internal finance. This is consistent with the *pecking order theory*.

FIGURE 13. Timeline



3.2. *** Nenov (2017, RFS).** This paper studies an *adverse selection* problem in a *general equilibrium* model. The paper shows *adverse selection* at credit market can give rise to *endogenous leverage* in financial intermediary.

3.2.1. *Settings.* There are two types of unit-mass agents, entrepreneurs and households. Entrepreneurs are endowed with e units of capital, and possess heterogeneous production technology: η fraction of them are endowed with high type (H) technology, the other $1-\eta$ fraction are endowed with low type (L) one..
The production technology of type-θ entrepreneur, $\theta \in \{H, L\}$, is given as followed:

$$y(k; p_H, R) = \begin{cases} Rk, & w.p. \quad p_H \\ 0, & w.p. \quad 1-p_H \end{cases}$$

$$y(k; p_L, R) = \begin{cases} Rk, & w.p. \quad p_L \\ 0, & w.p. \quad 1-p_L \end{cases}$$

Naturally we assume $p_H > p_L$.

A representative household is endowed with h units of capital and backyard production technology $f(k)$[2]. Entrepreneurs differ from households in having access to credit market, namely they can borrow at credit market to finance their production.

The timeline is shown in figure 13: At the beginning of period, state of aggregate and idiosyncratic productivity (R and p_θ) are realized. Entrepreneurs have private information on their own type, and engage themselves in purchase or resale of capital, borrowing or saving, subject to borrowing constraint and budget constraint (i.e. no equity injection condition). This model departs from existing ones in allowing entrepreneurs to optimally select financial contracts with promised repayment schemes backed by collateral goods. At the end of each period, production of investment projects are realized and capital goods fully depreciate. Debt contracts are executed and entrepreneurs consume the residual products and saving.

3.2.2. *Credit market.* Entrepreneurs borrow with standard debt contracts to finance their investment and capital acquisition, each characterized by its face value, β, and unit of collateral, k. Given the linear production technology and anonymity at the credit market, each type of corporate bond can also be characterized by unit face value (or face value to collateral ratio), denoted as $\omega \equiv \beta/k$. The repayment to each unit of debt $\omega < R$ issued by type θ, denoted as $d(\omega, \theta)$, is

$$d(\omega, \theta) = \begin{cases} \omega, & w.p. \quad p_\theta \\ 0, & w.p. \quad 1-p_\theta \end{cases}$$

[2]The neoclassical production technology features standard assumption that $f'(k) > 0$ and $f''(k) < 0$. We also assume $f'(h)$ is sufficiently small so that households always supply some capital at equilibrium.

Then the pricing equation of any debt contract indexed by ω and k, denoted as $b(\omega, k)$, can be derived from rational belief and no-arbitrage condition:

$$b(\omega, k) = E_\theta[d(\omega, \theta)k] \tag{56}$$

3.2.3. *Optimization Problem.* The optimization problem of H-type entrepreneurs is characterized as follows[3]:

$$v_H = \max_{k,s,\omega} p_H Rk - p_H \omega k + s \tag{57}$$

s.t.

$$k = e + i \geq 0$$

$$b(\omega, k) = E_\theta[d(\omega, \theta)k] \tag{58}$$

$$s = b(\omega, k) - qi \geq 0 \tag{59}$$

where i, s, q are net capital acquisition, saving and capital resale price respectively. Equation (58) is borrowing constraint from debt pricing equation in previous section, and equation (59) is budget constraint with implicit assumption of no equity injection.

If we denote $i_H(q)$ and $i_L(q)$ as policy function of capital acquisition for H-type and L-type entrepreneurs respectively, equilibrium resale price of capital can be pinned down by the following capital market clearing condition:

$$\eta i_H(q) + (1-\eta) i_L(q) = h - f'^{(-1)}(q)$$

where the right hand side is capital supply from households with neoclassical backyard production technology.

3.2.4. *First-Best.* Absent of any friction, it's immediate that low-type entrepreneurs do not produce and marginal products of capital are equalized across all producers (high-type entrepreneurs and households):

$$p_H R = f'(K^*_{nh}) \tag{60}$$

where K^*_{nh} denotes capital held by neoclassical households in planner's equilibrium. The aggregate size of capital held by H-type entrepreneurs, denoted as K^*_H, is thus

$$K^*_H = e + h - K^*_{nh} \tag{61}$$

3.2.5. *Financial Friction I: Exogenous Leverage.* In this section we incorporate a standard collateral borrowing constraint to an otherwise identical model without any friction. Due to limited commitment and costly liquidation, entrepreneurs can borrow up to ξ fraction of collateral value, so equation (58) becomes:

$$b(k) \leq \xi qk \tag{62}$$

and the lending standard applies to both high and low type entrepreneurs with access to credit market.

The implied investment policy rules of agents with endowed capital k, denoted as $i_\theta(k, R)$ for $\theta \in \{H, L, nh\}$, are

$$i_H(e, R) = \begin{cases} \frac{\xi}{1-\xi} e, & if \quad q \leq p_H R \\ 0, & otherwise \end{cases}$$

$$i_L(e, R) = \begin{cases} \frac{\xi}{1-\xi} e, & if \quad q \leq p_L R \\ 0, & otherwise \end{cases}$$

$$i_{nh}(h, R) = f'^{(-1)}(q) - h$$

[3]The parallel problems of low type entrepreneur and households are similar and left to appendix.

respectively. It's noticeable that conditional on that an entrepreneur makes positive net investment, his demand for capital is independent of asset price and productivity. These investment rules, along with capital market clearing condition:

$$\eta i_H(e,R) + (1-\eta) i_L(e,R) = i_{nh}(h,R) \tag{63}$$

define competitive equilibrium allocation with financial friction.

3.2.6. *Financial Friction II: Endogenous Leverage.* Despite parsimony of previous model in capturing frictions at financial market, the fashion has been criticized for lack of realism, say heavy reliance on exogenous shocks to generate credit and leverage cycles that are often perceived as endogenous outcome[4]. In the extended model we depart from that approach by introducing endogenous leverage arising from an information asymmetry problem between creditors and borrowers at financial market.

We assume productivity and thus repayment probability are private information to entrepreneurs. We start by considering a separating equilibrium where H-type entrepreneurs choose some debt contract (ω_H) as a signal to separate themselves from low quality borrowers[5]. An immediate result from this separation is that price of debt contract:

$$b(\omega_H, k) = p_H \omega_H k, \quad \text{for any } \omega_H \leq R$$

Given this fair pricing of financial contract at separating equilibrium, the individual rationality (IR) condition for H-type entrepreneurs holds for any $q \leq p_H R$:

$$(p_H R - p_H \omega_H) k \geq (q - p_H \omega_H) k \tag{64}$$

where expected return to equity (expected output minus repayment) on the left-hand-side is greater than internal investment made by entrepreneurs on the right-hand-side.

The incentive compatibility (IC) constraint, on the other hand, rules out any contract delivering positive profit to low type from mimicking high type:

$$p_L(R - \omega_H) \leq q - p_H \omega_H \tag{65}$$

and is binding at equilibrium[6]. By assumption low type entrepreneurs default with probability $1-p_L$, so the expected return to equity is $p_L(R-\omega)$ for each unit of capital. Incentive compatibility is guaranteed when expected return to equity is no greater than internal investment made by entrepreneurs on the right hand side. The IC constraint gives the maximum separating leverage ratio can be chosen by high type borrowers at separating equilibrium:

$$\omega_H \leq \frac{q - p_L R}{p_H - p_L}$$

[4]As Fostel and Geanakoplos (2014) puts, changing leverage is not main focus of credit cycle models; in fact, in those models leverage negatively comoves with asset prices, mitigating cycles in stead of driving them.

[5]Discussion on pooling equilibrium is omitted for two reasons: firstly, as Nenov (2016) and many have put, pooling equilibrium is difficult to sustain after simple belief refinements; secondly, in our context with linear production technology and risk-neutral preference, we can show separation is always preferred by high-type entrepreneurs. In last section we will discuss implication of regime switching when assumptions of linearity are relaxed.

[6]It can be shown that the incentive compatibility constraint can be alternatively derived from optimization problem and budget constraint of low-type entrepreneurs.

The equilibrium leverage ratio of debt contract, together with its pricing equation, delivers an endogenous borrowing limit for (H-type) entrepreneurs:

$$b(\omega_H, k) \leq p_H \frac{q - p_L R}{p_H - p_L} k \tag{66}$$

Compared with standard borrowing constraint where leverage (or loan-to-value) ratios are often exogenous, equation (66) shows that this borrowing constraint features an endogenous, more precisely, procyclical leverage with respect to price changes[7].

The procyclical property of leverage ratio here seems realistic and can be supported by findings of Brunnermeier (2009) and Geanakoplos (2010) etc. on countercyclical lending standards. Unlike those driven by heterogeneous belief or value-at-risk rules, the endogenous leverage cycle here is due to countercyclical adverse selection problem: At separating equilibrium low quality borrowers are indifferent between borrowing and not borrowing. When asset price decreases, the low type would strictly prefer to borrow if leverage ratio didn't decrease. Thus in response to lower asset prices, high quality borrowers lower leverage of chosen financial contract to maintain separation from low type borrowers, and by doing so they leave more "skin in the game".

Thus the model introduces a new channel of pecuniary externality through leverage ratio: low asset prices depress demand not only through discounted collateral values, but through a lower loan-to-value ratio as well. This novel yet realistic feature allows us to address many interesting topics, for example, self-fulfilling prophecy and in current context, solving the puzzles in capital reallocation literature.

After substituting equation (58) with equation (66), first-order conditions from optimization problem (57) imply the following capital market clearing condition that pins down equilibrium asset price of resale capital q:

$$\frac{p_H(q - p_L R)}{p_L(p_H R - q)} \eta e + (1-\eta)(-e) = h - f'^{(-1)}(q), \quad q \in [p_L R, p_H R] \tag{67}$$

Competitive Separating Equilibrium: The competitive separating equilibrium is defined as the sequence of endogenous variables $\{q, \omega_H, k_\theta, s_\theta\}$, $\theta \in \{H, L, nh\}$ that are consistent with first-order conditions of agents' optimization problems (in appendix) and
Equation (64): individual rationality condition of H-type;
Equation (65): incentive compatibility condition of L-type;
Equation (67): capital market clearing condition.

[7]To see this, we can rewrite equation (66) into a standard form:

$$b \leq \frac{p_H(q - p_L R)}{q(p_H - p_L)} qk \equiv \tilde{\xi} qk$$

where leverage ratio $\tilde{\xi} = \frac{p_H q - p_H p_L R}{q(p_H - p_L)}$ positively co-moves with asset prices. $1 - \tilde{\xi}$, often interpreted as down-payment ratio in leverage buyouts (LBOs), is time-varying and countercyclical.

3.3. *** Martin (2008, WP).** This paper (*Endogenous Credit Cycles*) studies an *adverse selection* problem at credit market in a *general equilibrium* model. The paper shows *adverse selection* at credit market can give rise to *endogenous credit cycles* and *endogenous business cycles* without introducing any shock.

3.3.1. *Introduction.* motivation of the paper

- large literature: financial markets and macroeconomic fluctuations
  - financial system as amplifier of exogenous shocks
  - lax credit and rapid expansion of output as the seeds of a future downturn ?
- this research: financial market as a source of macroeconomic fluctuations
  - exhibition of fluctuations absent of exogenous shock
  - endogenous boom-bust cycles

highlight of the paper

- adverse selection
  - borrowers with private information: good or bad
  - credit contract under asymmetric information
  - endogenous change of lending standards as source of fluctuations
- net worth and lending standard
  - higher net worth $\Leftrightarrow$ more investment
  - low net worth $\Rightarrow$ costly separation $\Rightarrow$ pooling contract
  - high net worth $\Rightarrow$ easier separation $\Rightarrow$ separating contract
- regime switch and fluctuation
  - low net worth $\Rightarrow$ pooling contract $\Rightarrow$ higher investment $\Rightarrow$ higher net worth $\Rightarrow$ separating contract $\Rightarrow$ lower investment $\Rightarrow$ low net worth

stylized facts

- procyclical net worth and endogenous reversion into recession
- lending standard over the business cycles

3.3.2. *Set-up.* The model features overlapping generations that live two periods: young and old. A new generation of measure one is born at every period. The utility goal of the young is to maximize expected old-age consumption of final goods. The young is endowed with one unit of labor and supply it inelastically. Their save their labor income in the production of capital goods. The old own the capital stock and live off their capital income. In each period, the labor supplied by the young is combined with capital owned by the old to produce final consumption goods according to a constant-return-to-scale technology. Capital fully depreciates after utilization.

- production technology of final product:

$$y_t = \theta g(k_{t-1}, 1) \tag{68}$$

- wage received by the young:

$$w_t(k_{t-1}) = \theta[g(k_{t-1}) - k_{t-1}g'(k_{t-1})] \tag{69}$$

- capital gain received by the old

$$q_t(k_{t-1}) = \theta g'(k_{t-1}) \tag{70}$$

Capital. Only a fraction of the young population, referred to as entrepreneurs, have the technology to produce capital. Entrepreneurs invest in production of capital when they are young, and consume capital gain when they are old. There are two types of entrepreneurs, a measure of λ^G being good (G) and λ^B being bad (B). We assume $\lambda^G + \lambda^B < 1$, and measure $1 - \lambda^G - \lambda^B$ are households. The technology is heterogeneous in the following way:

- The investment made by the young entrepreneur can either succeed or fail in subsequent period. The probability of success is p^j, and we assume $p^G > p^B$.
- When the investment succeeds, an entrepreneur who invests I unit of consumption good receive $\alpha^j f(I)$ units of capital [8]. We assume $\alpha^G < \alpha^B$.
- When the investment fails, an entrepreneur receives nothing. We assume $p^G\alpha^G > p^B\alpha^B$.

Credit Market and Financial Contract. There is natural demand for credit market in this economy: while entrepreneurs can invest their wage income directly into production of capital, household needs to save their income. At the meantime, entrepreneurs may need external source to finance their investment.

- financial intermediary: competitive and risk neutral banks who take deposit with promised gross interest r_t.
- loan contract is characterized by (I_t, R_t, c_t), where
  - I_t: amount of consumption goods lent to borrower
  - R_t: gross interest rate on the loan
  - c_t: percentage of the loan that entrepreneur save as collateral using their own wealth.
- state-contingent repayment and default outcomes
  - success: entrepreneur repays R_tI_t and claims residual value of project
  - failure: bank takes collateral plus interest rate and take residual value (=0)
- expected profit of entrepreneur j:

$$\pi^j(I_t, R_t, c_t) = r_t w_t + p^j[q^e_{t+1}\alpha^j f(I_t) - R_tI_t] - (1-p^j)r_tc_tI_t \tag{71}$$

- expected profit of bank from the contract:

$$\pi^b(I_t, R_t, c_t) = p^j R_t I_t + (1-p^j)r_tc_tI_t - r_tI_t \tag{72}$$

3.3.3. *First-Best: Full Information.* Here in this section we discuss main property of loan contract in a partial equilibrium setting, where deposit interest rate r and expected rental price of capital q^e are taken as given. Under full information, the equilibrium $\{I_t^{j*}, R_t^{j*}, c_t^{j*}\}$ is straightforward:

- optimal size of funding I_t^{j*}:

$$f'(I_t^{j*}) = \frac{r}{q^e\alpha^j p^j} \quad \text{for j=G,B} \tag{73}$$

- collateral required by banks c_t^{j*} and gross interest rate R_t^{j*}:

$$p^j R_t^{j*} + (1-p^j)c_t^{j*} r = r \quad \text{for j=G,B} \tag{74}$$

Thus under full information, 1) good entrepreneurs invest more than bad entrepreneurs; 2) banks break even, and 3) investment is independent of entrepreneurs' wealth w_t.

[8]The function $f()$ is an increasing, concave, and satisfies Inada condition.

- i.e., gross interest rate R_t^{j*} (not unique) if $w_t = 0$:

$$R_t^{j*} = \frac{r}{p^j} \quad \text{for j=G,B} \tag{75}$$

3.3.4. *Asymmetric Information.* Now consider the case of asymmetric information, where *ex ante* banks are unable to distinguish among different types of borrowers. Following Hellwig (1987), contract at credit market is modelled in three stages:

- 1st stage: banks design contract;
- 2nd stage: entrepreneurs apply for these contract;
- 3rd stage: banks accept or reject applications.

We assume exclusivity and no cross-subsidization that 1) entrepreneurs can apply to no more than one contract; 2) banks are not allowed to offer contracts that lose money in expectation. The following equilibrium contracts are characterized for an economy indexed by $\{r, q^e, w_t\}$.

3.3.5. *Asymmetric Information I: Separating Equilibrium: $C^{SEP}(r, q^e, w_t)$.* **Definition 1**: Given $\{r, q^e, w_t\}$, a separating equilibrium is characterized by contracts $\{(I_t^G, R_t^G, c_t^G), (I_t^B, R_t^B, c_t^B)\}$ that satisfy the following constraints:

- feasibility:

$$c_t^j \in [0, \frac{w_t}{I_t^j}] \quad \text{for j=G,B} \tag{76}$$

- incentive compatibility:

$$\pi^j(I_t^j, R_t^j, c_t^j) \geq \pi^j(I_t^i, R_t^i, c_t^i) \quad \text{for } i \neq j \text{ and } i, j \in \{G, B\} \tag{77}$$

- break-even condition for banks:

$$p^j R_t^j + (1 - p^j) c_t^j r = r \quad \text{for j=G,B} \tag{78}$$

- no deviation for banks.

Proposition 1: Given $\{r, q^e, w_t\}$, a separating equilibrium is characterized by contracts $\{(I_t^G, R_t^G, c_t^G), (I_t^B, R_t^B, c_t^B)\}$ that satisfy:

- contract chosen by the bad-type is not distorted:

$$(I_t^B, R_t^B, c_t^B) = (I_t^{B*}, R_t^{B*}, 0) \tag{79}$$

- contract chosen by the good-type is distorted[9]:

$$c_t^G = \frac{[q^e p^B \alpha^B f(I_t^G) - \frac{p^B}{p^G} I_t^G r] - [q^e p^B \alpha^B f(I_t^B) - I_t^B r]}{(1 - \frac{p^B}{p^G}) I^B r} \leq 1 \tag{80}$$

$$q^e \alpha^G p^G f'(I_t^G) > r \quad \Rightarrow \quad c_t^G = \frac{w_t}{I_t^G} \tag{81}$$

[9]The good type solves the following problem:

$$\max_{I^G, R^G, c^G} \pi^G \equiv rw + p^G[q^e \alpha^G f(I^G) - R^G I^G] - (1 - p^G) c^G I^G r$$

s.t.

$$p^G R^G + (1 - p^G) c^G r = r = p^B R^B$$
$$p^B[q^e \alpha^B f(I^B) - R^B I^B] = p^B[q^e \alpha^B f(I^G) - R^G I^G] - (1 - p^B) c^G I^G r$$
$$c^G \in [0, \frac{w}{I^G}]$$

Proposition 1 implies that cost of separation is undertaken by good-type entrepreneurs who either provide higher level of collateral or choose lower level of investment, aka lower leverage.

- collateral: a costless way of screening / separating entrepreneurs.
  - good-type entrepreneurs are willing to increase c^G to lower R^G
  - bad-type entrepreneurs are worse off
  - separation in this way becomes very costly when w_t is low [10]
  - increase in w_t enhances the probability of separation via collateralization.
  - for sufficiently high w_t, first-best can be achieved: $I_t^G = I_t^{G*}$

The equilibrium doesn't always entail separation. A pooling equilibrium may exist whenever it Parato dominates the separating contract of *Proposition 1*.

3.3.6. *Asymmetric Information II: Pooling Equilibrium:* $C^{POOL}(r, q^e, w_t)$. **Definition 2**: Given $\{r, q^e, w_t\}$, a pooling equilibrium is characterized by contracts $\{(\bar{I}_t, \bar{R}_t, \bar{c}_t)\}$ that satisfy the following constraints:

- feasibility:

$$\bar{c}_t \in [0, \frac{w_t}{\bar{I}_t}] \tag{82}$$

- break-even condition for banks:

$$E_j[p^j \bar{R}_t + (1 - p^j)\bar{c}_t r] = r \tag{83}$$

- no deviation for banks.

Proposition 2: Given $\{r, q^e, w_t\}$, a pooling equilibrium $\{(\bar{I}_t, \bar{R}_t, \bar{c}_t)\}$ satisfies [11]:

- gross interest rate

$$\bar{R}_t = r\frac{1 - (1 - \bar{p})\bar{c}_t}{\bar{p}} \tag{84}$$

- collateral requirement

$$\bar{c}_t = \frac{w_t}{\bar{I}_t} \tag{85}$$

- investment size

$$p^G \alpha^G f'(\bar{I}_t) = \frac{p^G}{\bar{p}}\frac{r}{q^e} \tag{86}$$

Proposition 2 implies the following properties of pooling equilibrium:

- investment size is independent of wealth w_t
- collateral constraint is binding and is increasing with wealth w_t
- degree of cross-subsidization is decreasing with wealth w_t

[10] It can be shown that when $w_t = 0$, $I_t^G < I_t^B$ (inefficiency).

[11] At pooling equilibrium good-type entrepreneurs solve the following problem:

$$\max_{\bar{I}, \bar{c}} \pi^G \equiv rw + p^G[q^e \alpha^G f(\bar{I}) - \bar{R}\bar{I}] - (1 - p^G)r\bar{c}\bar{I}$$

s.t.

$$\bar{p}\bar{R} + (1 - \bar{p})\bar{c}r = r$$

$$0 \le \bar{c}$$

$$\bar{c} \le \frac{w}{\bar{I}}$$

3.3.7. *Equilibrium Contract:* $C^{EQ}(r, q^e, w_t)$.

- Separating or pooling contract?
  - Depend on the level of wealth w_t
  - when w_t is low, separation is costly: i.e., for $\bar{p} > \frac{\alpha^B p^B}{\alpha^G}$, the equilibrium is always pooling when $w_t = 0$.
  - when w_t is increased, the separating equilibrium emerges.
  - cut-off for regime switch: $w^*(r, q^e)$
- what's the impact of regime switch on aggregate investment?
  - investment drops as long as $\bar{p} > \frac{\alpha^B p^B}{\alpha^G}$
  - when good-type is abundant, i.e. $\bar{p} > \frac{\alpha^B p^B}{\alpha^G}$, pooling equilibrium represents mostly technology of the good-type: $\bar{I}_t(r, q^e) > I_t^B(r, q^e)$
  - switch to separation contracts investment made by bad-type (obvious).
  - switch to separation contracts investment made by good-type at the margin [12]
  - aggregate investment is discontinuous at the switching point:

$$\bar{I}_t(r, q^e) > I_t^G(r, q^e, w^*) > w^* > I_t^B(r, q^e, w^*)$$

3.3.8. *Endogenous Cycles.*

- timeline
  - investment project undertaken by the old yields capital stock of the economy;
  - production of final goods takes place using capital and labor supplied by the young
  - the old repay their debt; the young save their labor income and invest.
- assumptions
  - unique, stable steady state at full information
  - parameter: $\bar{p} > \frac{\alpha^B p^B}{\alpha^G}$
  - exogenous interest rate: r

Definition 3: *intertemporal equilibrium* of the asymmetric information economy is defined as a trajectory $\{k_t, w_t, q_{t+1}^e, r_t, C^{EQ}(w_t, q_{t+1}^e) : t \geq 0\}$ that satisfies

- contract $C^{EQ}(w_t, q_{t+1}^e)$ as characterized before
- labor and capital market clears: w_t and q_t
- perfect foresight: $q_{t+1}^e = q_{t+1}$

3.3.9. *Full Information: No Dynamics.* The equilibrium under full information is trivial as in section 3.3.3, where

- optimal size of funding I_t^{j*} independent of state variables:

$$\alpha^j p^j f'(I_t^{j*}) = \frac{r}{q_{t+1}^e} \quad \text{for j=G,B}$$

- perfect foresight:

$$q_{t+1}^e = q_{t+1} = \theta g'[k_t^*(r_t, q_{t+1}^e)] \tag{87}$$

- capital stock $k_t^*(r_t, q_{t+1}^e)$ independent of state variables:

$$k_t^*(r_t, q_{t+1}^e) = \lambda^G \alpha^G p^G f[I_t^{G*}(r_t, q_{t+1}^e)] + \lambda^B \alpha^B p^B f[I_t^{B*}(r_t, q_{t+1}^e)] \tag{88}$$

[12] *Prima facie* this result is surprising. The intuition is as follows: Good-type entrepreneurs are indifferent between pooling and separating equilibrium at the switching point. Given that pooling contract provides fund at a higher cost due to cross-subsidization, it must entail higher level of investment compared to separating equilibrium. In other words, $\bar{I}_t(r, q^e) > I_t^G(r, q^e, w^*)$.

Applying assumption 1 regarding existence and uniqueness of steady state, the economy always converges to a unique equilibrium denoted as $\{k^*, w^*, q^*\}$.

3.3.10. *Pooling Regime: No Dynamics.* The consideration of pooling regime under asymmetric information resembles that under full information, where

- optimal size of funding I_t^{j*} independent of state variables:

$$\alpha^j p^j f'(\bar{I}_t) = \frac{r}{q_{t+1}^e} \frac{p^G}{\bar{p}}$$

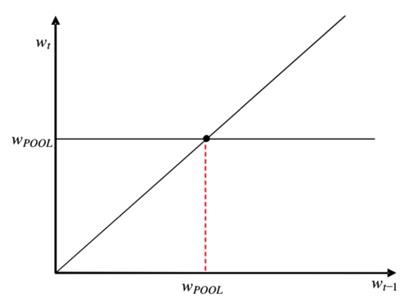
- perfect foresight:

$$q_{t+1}^e = q_{t+1} = \theta g'[k_t^{POOL}(r_t, q_{t+1}^e)] \tag{89}$$

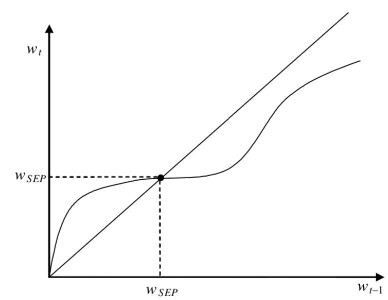
- capital stock $k_t^{POOL}(r_t, q_{t+1}^e)$ independent of state variables:

$$k_t^{POOL}(r_t, q_{t+1}^e) = [\lambda^G \alpha^G p^G + \lambda^B \alpha^B p^B] f[\bar{I}_t(r_t, q_{t+1}^e)] \tag{90}$$

Applying assumption 1 regarding existence and uniqueness of steady state, there is unique and stable steady state in the pooling regime as well. We denote this unique equilibrium as $\{k^{POOL}, w^{POOL}, q^{POOL}\}$. Similar to full information regime, for any w_{t-1} in the pooling equilibrium, $w_t = w^{POOL}$ so that the economy jumps to the steady state regardless of initial condition, i.e. there are no dynamics in the pooling regime.



Wage Dynamics under Pooling Contracts



Wage Dynamics under Separating Contracts

3.3.11. *Separating Regime Dynamics.* Contrary to previous two regimes, there are dynamics in the separating regime: higher $w_t \Rightarrow$ higher investment $\Rightarrow$ higher w_{t+1}. In the separating regime,

- level of investment $I_t^{B,SEP}(r_t, q_{t+1}^e)$ independent of w_t:

$$\alpha^B p^B f'(I_t^{B,SEP}) = \frac{r}{q_{t+1}^e}$$

$I_t^{G,SEP}(r_t, q_{t+1}^e, w_t)$ dependent on w_t

$$\frac{w_t}{I_t^{G,SEP}} = \frac{[q^e p^B \alpha^B f(I_t^{G,SEP}) - \frac{p^B}{p^G} I_t^{G,SEP} r] - [q^e p^B \alpha^B f(I_t^{B,SEP}) - I_t^{B,SEP} r]}{(1 - \frac{p^B}{p^G}) I^{B,SEP} r}$$

- perfect foresight:

$$q_{t+1}^e = q_{t+1} = \theta g'[k_t^{SEP}(r_t, q_{t+1}^e, w_t)] \tag{91}$$

- capital stock $k_t^{SEP}(r_t, q_{t+1}^e, w_t)$:

$$k_t^{SEP}(r_t, q_{t+1}^e, w_t) = \lambda^G \alpha^G p^G f[I_t^{G,SEP}(r_t, q_{t+1}^e, w_t)] + \lambda^B \alpha^B p^B f[I_t^{B,SEP}(r_t, q_{t+1}^e)] \tag{92}$$

The economy might display unique, stable steady state or multiple steady states. Here we restrict our attention to the former and denote the economy as $\{k^{SEP}, w^{SEP}, q^{SEP}\}$.

Assumption:

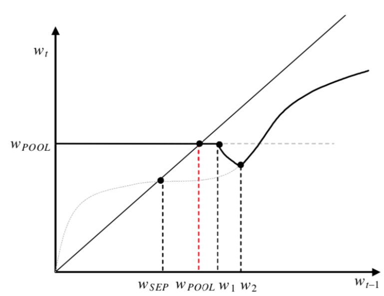
$$w^{SEP} < w^{POOL} \tag{93}$$

3.3.12. *Regime Switching and Cycles.* **Proposition 3**: Assume an economy in which $\bar{p} > \frac{\alpha^B p^B}{\alpha^G}$. For wage $w_t \in [0, \bar{w}]$, there exists a unique pair of switching wages (w_1, w_2) such that:

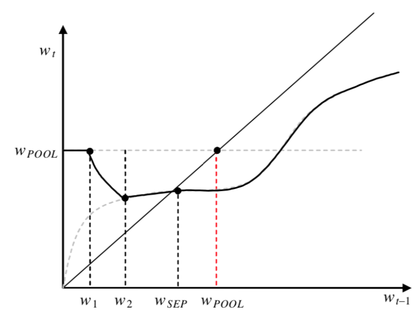
- if $w_t \leq w_1$, then the equilibrium loan contracts at time t are pooling;
- if $w_t \geq w_2$, then the equilibrium loan contracts at time t are separating;
- if $w_1 \leq w_t \leq w_2$, then the equilibrium loan contracts at time involve randomization between pooling and separating contracts.

Now we proceed to consider the following cases:

- Case 1: $w^{SEP} < w^{POOL} \leq w_1$:
  - unique, stable steady state at w^{POOL}
  - oscillatory convergence
  - monotonic convergence for initial $w_0 < w_1$
  - convergence with overshooting for some initial $w_0 > w_1$
- Case 2: $w_2 \leq w^{SEP} < w^{POOL}$:
  - unique, stable steady state at w^{SEP}
  - oscillatory convergence
  - monotonic convergence for initial $w_0 > w_2$
  - convergence with overshooting for some initial $w_0 < w_2$



Wage Dynamics under Case 1: $w_{POOL} \leq w_1$

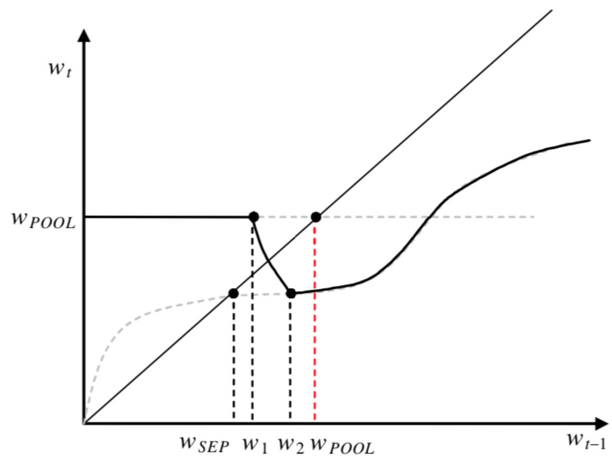


Wage Dynamics under Case 2: $w_{SEP} \geq w_2$

- Case 3: $w_1 < w^{POOL}; w^{SEP} < w_2$:
  - unique steady state at w^{SEP}
  - unstable steady state: permanent fluctuation
  - stable steady state: convergence with fluctuation

The last case is of particular interest: an economy with no dynamic under full information displays fluctuation in the presence of adverse selection [13]. The intuition is straightforward: For low level of w_t, separation is costly so that the economy is at pooling regime where investment and wages gradually build up. When the increase is wealth is sufficiently large, the economy switches to equilibrium with partial or complete separating contracts, and consequently, a fall in output. The decrease in output, in turn, decreases entrepreneurs' wealth and the economy goes back to pooling regime. In that sense, the economy features endogenous cycle without introduction of exogenous shock.

[13] *Proposition 4* proves existence case 3 in any economy satisfying case 1 and 2.



Wage Dynamics under Case 3: $w_{SEP} < w_2$ and $w_{POOL} > w_1$

3.3.13. *Conclusion.*

- implication 1: financial friction
  - investment is increasing with net worth at separating regime
  - investment is independent of net worth at pooling regime
  - investment is more sensitive to net worth at recession (Bernanke et al., 1999)
- implication 2: bank lending standard
  - changes in lending standards are determined by economy activity (wealth)
  - changes in lending standards are determinant of economy activity (investment)
  - procyclical loan size and countercyclical rates of collateralization
  - "lax" lending standard associated with low variance of interest rate (pooling)
  - "tight" lending standard associated with high variance of interest rate (separating)
- implication 3: positive productivity shock
  - net worth increases ⇒ aggregate investment increases (amplification)
  - aggregate savings increase ⇒ aggregate investment decrease (mitigation)
  - closed economy vs. open economy
  - financial liberalization and macroeconomic stability
- implication 4: sources of fluctuation
  - no aggregate shock
  - adverse selection ⇒ changes in lending standard
  - perfect competition in credit market
- future directions:
  - OLG ⇒ infinite horizon: endogenize interest rate r
  - liquidity and macroeconomy (Taddei, 2010)
  - endogenize distribution of different types: extensive margin problem (Hu, 2017) (Fishman et al., 2019)

3.4. *** Kurlat (2013, AER).** This paper studies an *adverse selection* problem in a *dynamic general equilibrium* model. This paper shows that asymmetric information about quality of assets used for intertemporal trade can amplify aggregate shocks and generate cyclical frictions distorting investment, financing and reallocation decision.

3.4.1. *Set up.*
Agents. two types:

- workers (mass L)
  - supply labor inelastically
  - no access to financial market
  - " hand-to-mouth" consumers: c_t^w
- entrepreneurs (mass 1)
  - heterogeneous indexed by j
  - log-utility in consumption: c_t^j
  - not working

Technology. capital + labor → consumption goods

- constant return to scale:

$$Y_t = Y((1-\lambda)K_t, L; Z_t) \tag{94}$$

- exogenous productivity: Z_t
- capital: $K_t = \int k_t^j dj$
  - accumulated from investment project: i_t^j
  - investment opportunity: r.v. $A_t^j \sim F(A)$, with support over $[A^{min}, A^{max}]$
  - investment opportunity A_t^j is private information to entrepreneur j
  - projects owned by entrepreneur j: k_t^j
  - λ projects become lemon (useless) in each period
  - $1-\lambda$ non-lemon projects turn to γ projects next period

$$K_{t+1} = \gamma(1-\lambda)K_t + \int i_t^j A_t^j dj \tag{95}$$

Information. The quality of period-t project is private information to entrepreneur at period t, but becomes public information at t+1.

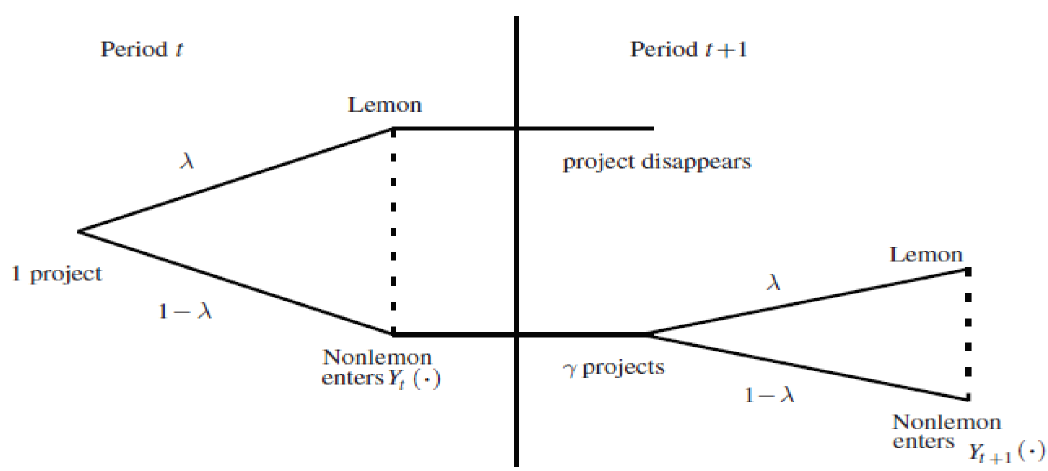


FIGURE 1. INFORMATION ABOUT A PROJECT OVER TIME

Notation. . The aggregate state variable can be summarized as $X \equiv \{z, \bar{K}\}$, with transition probability $\rho(X, X')$. We denote w as wage paid to workers. We denote r to be return on project. p_L and p_{NL} denote price of lemon and non-lemon project. d_L and d_{NL} denote quantity of lemon and non-lemon project purchased. Each unit of state-contingent security $b(X')$ promise one unit of consumption good in state X'.

3.4.2. *Scenario I: First-Best.* In the first-best scenario, information is public (symmetric) and there are complete markets (for borrowing and lending). This immediately implies that $p_L = 0$, and $d_L = 0$.

Entrepreneurs solve

$$V(k, b, X) = \max_{c,k',i,d_{NL},b(X')} \left[u(c) + \beta \mathbb{E}\left[V\left(k', b\left(X'\right), X'\right) \mid X\right]\right] \tag{96}$$

s.t. a budget constraint

$$c + i + p_{NL}(X) d_{NL} + \mathbb{E}\left[\rho\left(X, X'\right) b\left(X'\right)\right] \leq r(X)(1-\lambda)k + b \tag{97}$$

law of motion

$$k' = \gamma\left[(1-\lambda)k + d_{NL}\right] + A^j(X) i \tag{98}$$

$$i \geq 0, \quad d_{NL} \geq -(1-\lambda)k \tag{99}$$

The allocation at first-best is straightforward: entrepreneur with best investment opportunity ($A^j = A^{max}$) is the only one undertaking investment projects. He finances this investment by issuing claims to consumption goods one period ahead to other entrepreneurs. State-contingent security is used to insure against aggregate risk. The value of consumption good is priced at A^{max}.

3.4.3. *Scenario II: Symmetric Information with Borrowing constraint.* In this scenario, while information is symmetric (which again implies that $p_L = 0$, and $d_L = 0$), entrepreneurs cannot borrow against future wealth, i.e. $b(X') > 0$. No borrowing implies no lending in equilibrium. Now entrepreneurs solve previous problem with an additional constraint:

$$b(X') = 0 \tag{100}$$

Now, the only way to achieve intertemporal trade is by selling project. We also assume only built-up project can be traded.

The allocation in this scenario is as follows: entrepreneur with investment opportunity below a threshold ($A^* = \frac{\gamma}{p_{NL(X)}} < A^{max}$) will not undertake investment projects. Those above threshold will sell all non-lemon project ("cash cow") to obtain consumption good and make new investment. The value of consumption good is priced by marginal buyers with $A^j = A^*$.

3.4.4. *Scenario III: Asymmetric Information.* In this scenario, we go back to our initial assumption at only the owner knows the quality of his project and observes investment opportunity A^j. Rational expectation is assumed throughout the model that trading price reflects true value of λ^M, proportion of lemons sold at the market. We maintain the assumption that no intertemporal borrowing or lending is available but selling individual projects. Now entrepreneurs solves a new problem as:

$$V(k, A, X) = \max_{c,k',i,s_L,s_{NL},d} \left[u(c) + \beta \mathbb{E}\left[V\left(k', A', X'\right) \mid X\right]\right] \tag{101}$$

s.t.

$$c + i + p(X)\left[d - s_L - s_{NL}\right] \leq r(X)(1-\lambda)k \tag{102}$$

$$k' = \gamma\left[(1-\lambda)k + \left(1-\lambda^M(X)\right)d - s_{NL}\right] + Ai \tag{103}$$

$$i \geq 0, \quad d \geq 0 \tag{104}$$

$$s_L \in [0, \lambda k], \quad s_{NL} \in [0, (1-\lambda)k] \tag{105}$$

Decision rules. We start by considering buying, selling and investment decisions of entrepreneurs.

- as long as $p > 0$, the entrepreneurs will sell all their lemon project: $s_L = \lambda k$
- return (t+1 project) to buying project (portfolio with lemon and non-lemon): $\frac{\gamma(1-\lambda^M)}{p}$
  we denote this as A^M, market rate of return
- cost of t+1 project to sell for one consumption good: $\frac{\gamma}{p}$
  it holds that $\frac{\gamma}{p} > A^M$
- return to investment: A^j

The optimal decision is characterized in figure below:

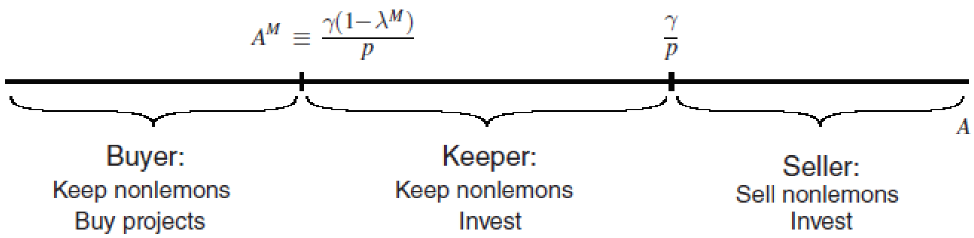


Figure 2. Buying, Selling, and Investing Decision as a Function of A

which implies the following decision rules

	if Buyer: $A \in [0, A^M]$	if Keeper: $A \in \left(A^M, \frac{\gamma}{p}\right]$	if Seller: $A \in \left(\frac{\gamma}{p}, \infty\right)$
$s_L =$	λk	λk	λk
$d =$	$\max\left\{\frac{k' - \gamma(1-\lambda)k}{\gamma(1-\lambda^M)}, 0\right\}$	0	0
$s_{NL} =$	$\max\left\{\frac{\gamma(1-\lambda)k - k'}{\gamma}, 0\right\}$	$\max\left\{\frac{\gamma(1-\lambda)k - k'}{\gamma}, 0\right\}$	$(1-\lambda)k$
$i =$	0	$\max\left\{\frac{k' - \gamma(1-\lambda)k}{A}, 0\right\}$	$\frac{k'}{A}$

3.4.5. *Adding Aggregate Shock.*

Assumption 1. We make an additional assumption here that $A^m(p)$ is decreasing. In general, the market return $A^m(p)$ can be either increasing or decreasing in p. An increase in the price has a direct effect of lowering returns by making projects more expensive and an indirect effect of improving returns by increasing the proportion of entrepreneurs who choose to sell their non-lemons. Assumption 1 helps eliminate discussion on multiple solution.

Productivity Shock. If in equilibrium price of project is positive, $p^* > 0$, then a positive productivity shock on Z leads to

- a higher price of projects,
- lower market returns for Buyers (if Assumption 1 holds),
- a lower proportion of lemons in the market, and
- higher capital accumulation (if Assumption 1 holds).

With asymmetric information,

- the price of projects increases more
- the market returns for Buyers fall less, and
- capital accumulation increases more (for λ small enough) .

Investment Shock. If in equilibrium price of project is positive, $p^* > 0$, then a positive investment shock on F(A) leads to

- ambiguous effect on price of projects,
- higher market returns for Buyers (if Assumption 1 holds),
- a lower proportion of lemons in the market, and
- higher capital accumulation (if Assumption 1 holds).

With asymmetric information,

- the price of projects falls less
- the market returns for Buyers increases more
- capital accumulation increases more (for λ small enough) .

4. Information Asymmetry III: Interaction b/w Adverse Selection and Moral Hazard

4.1. *** Dewatripont and Maskin (1995, REStud).** This paper shows that under *adverse selection*, unprofitable project may still be financed. This paper also shows that *ex post moral hazard* problem may lead to *ex ante adverse selection* problem.

4.1.1. *Settings.* There are three periods, one entrepreneur, and either one or two creditors (bank). The entrepreneur's project can be good or poor. A good project is completed after one period; a poor project requires two periods to complete. Whether good or poor, the project requires one unit of capital investment per period.

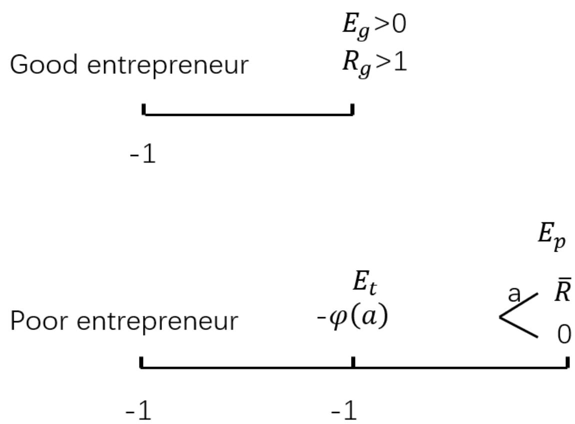


Figure 14. Timeline

Good Project: The private benefit of good project for entrepreneur is E_g ($E_g \geq 0$). And a good project generates return $R_g > 1$.
Poor Project: E_t is the entrepreneur's benefit of a poor project when her project is terminated after the first period, whereas E_p is her benefit from a complete project. $E_p >> E_t$.
Bank. Bank doesn't know entrepreneur's type at the first period (information asymmetry).
If the project is good, bank's payoff is $R_g - 1$.
If the project is poor, bank obtains nothing unless he agrees to refinancing at the beginning of the second period.
The project return at end of the second period is a random variable that is either 0 or $\overline{R}$. Bank can monitor the project at the beginning of period 2 and it can influence the distribution of R through monitoring. Suppose the effort is a and it is also the probability of $\overline{R}$. The cost of bank's effort is $\varphi(a)$.

4.1.2. *Optimal Effort.* The bank's optimization problem at date 1 is:

$$\max_a \overline{R}a - \varphi(a) \tag{106}$$

First-order condition implies an optimal effort at a^* such that

$$\overline{R} = \varphi'(a^*) \tag{107}$$

So the expected payoff for bank is $\Pi_p^* = \overline{R}a^* - \varphi(a^*)$.

Payoffs under centralization

	Good project (assuming $E_g > 0$)	Poor project without refinancing	Poor project with refinancing
Entrepreneur	E_g	E_t	E_p
Bank	$R_g - 1$	-1	$\Pi_p^* - 2$

FIGURE 15. Equilibrium in Centralized Credit Market

4.1.3. *Centralized Credit Market.* Suppose we are in a centralized economy where there is only *one* bank. We assume $\Pi_p^* + E_p < 2$, $\Pi_p^* > 1$. In this case, at the beginning of period 2, the bank will still refinance the poor project since $\Pi_p^* - 2 > -1$, through the project has negative NPV ($\Pi_p^* - 2 + E_p < 0$).

4.1.4. *Decentralized Credit Market.* Now we assume there are *two* banks and each only have one unit of capital. In this case, if the project is good, the analysis will be the same. If the project is poor, it cannot be financed by original bank, denoted as B_1, at the beginning of period 2 since B_1 has no capital left. So the entrepreneur must turn to the other bank, B_2. To convince B_2 to loan a second unit of capital in the poor project, B_1 must give B_2 some shares of return, denoted as $\tilde{R_p}$. B_2's expectation of $\tilde{R_p}$ is based on its expectation of B_1's effort. The higher B_2's expectation of B_1's monitoring effort in period 1, the smaller this share can be. Let $\hat{a}$ be B_2's assessment of the expected level of B_1's monitoring activity. Then to induce B_2 to participate, Bank 2 should receive $1/\hat{a}$ when $\tilde{R_p} = \overline{R_p}$. This is because B_2 break even with no bargaining power. The IR condition implies that the investment by Bank 2, 1 dollar, should be equal to the expected gain $\hat{a} * \frac{1}{\hat{a}}$.
Taken above into consideration, B_1 chooses a to maximize

$$\max_a (\overline{R_p} - \frac{1}{\hat{a}})a - \varphi(a) \tag{108}$$

In equilibrium, $\hat{a} = a^{**}$ such that the following F.O.C is satisfied

$$\overline{R_p} = \varphi'(a^{**}) + 1/a^{**} \tag{109}$$

Recall from equation (107) that under centralization $\overline{R} = \varphi'(a^*)$, we get $a^{**} < a^*$ since B_1 gives part of the marginal return of monitoring to B_2. Therefore, $\Pi_p^{**} = \overline{R}a^{**} - \varphi(a^{**}) < \Pi_p^*$. In this case, B_1 not only has to pay one unit of return to B_2 at the end of period 2, but also receives lower payoff due to less effort exerted.
With less payoff for B_1 ($\Pi_p^{**} < \Pi_p^*$), if $\Pi_p^{**} < 1$, the poor project will be terminated at the end of period 1. So the negative NPV project is less likely to get refinanced. Knowing this, the poor project is less likely to start at first period.

Payoffs under decentralization

	Good project (if $E_g > 0$)	Poor project with no refinancing	Poor project with refinancing
E	E_g	E_l	E_p
B_1	$R_g - 1$	-1	$\Pi_p^{**} - 2$
B_2	0	0	0

Figure 16. Equilibrium in Decentralized Credit Market

4.1.5. *Conclusion.* The initial project selection is an adverse selection problem for entrepreneur and refinancing is a moral hazard problem for bank. The adverse selection problem of initial project has also been discussed in Stiglitz and Weiss (1981) in which credit rationing is a way to deal with this problem. This model is relevant to the soft budget constraint in centralized economy. The "softness" arises from the profitability of refinancing poor project. If the bank is very liquid, there will be soft budget problems. In a decentralized bank market, the budget constraint can be hardened, as we can see $\Pi_p^{**} < \Pi_p^*$ and poor projects are less likely to be refinanced.

4.2. *** Parlour and Plantin (2008, JF).** This paper studies endogenous degree of liquidity at secondary loan market under information asymmetry. This paper also shows that *ex post adverse selection* (bank to the secondary market) may cause *ex ante moral hazard* (firm shirking or bank not monitoring) problem.

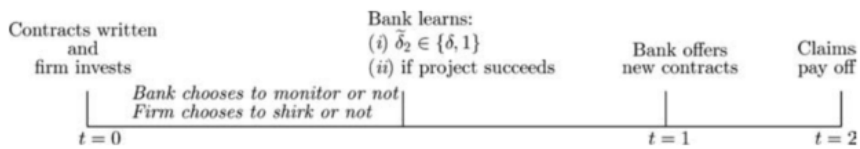


Figure 17. Timeline

4.2.1. *Settings.* Timeline: three-date (static) model

- date 0: Firm offers a contract to bank/investor, and invests in a project.
- date 1: Bank will decide to offer a new contract or not based on its information of the firm's project and outside opportunity.
- date 2: Claims payoff.

Type of agents:

- Firm: it has a two-period project with \$1 investment at $t = 0$. At $t = 2$ payoff is R with probability p, or 0 with probability $1 - p$.
- Outside investor: By raising fund from outside investors, the firm may shirk and get a private benefit B_F at $t = 1$. In this case, project will fail and $p = 0$.
- Bank: By active monitoring from the bank, the firm's private benefit reduces to b_F when shirk. When firm shirks, project will fail and $p = 0$. If bank does not monitor, it can get a private benefit B_B. Bank can acquires private information about the project at $t = 1$.

Important assumption: Bank maximizes utility function with a stochastic discount factor.

- At $t = 0$, the utility of a bank is

$$U^B(c_0, \tilde{c_1}, \tilde{c_2}) = E(c_0 + \delta_1 \tilde{c_1} + \delta_1 \tilde{\delta}_2 \tilde{c_2}) \tag{110}$$

  Where $\delta_1 \in (0, 1)$ and $\tilde{\delta}_2$ is a two-point random variable[14] whose realization at date 1 is

$$\tilde{\delta}_2 = \begin{cases} \delta \in (0,1) & \text{with prob} \quad q \\ 1 & \text{with prob} \quad 1-q \end{cases}$$

[14] $\tilde{\delta}_2$ proxies for unanticipated changes in the opportunity cost of carrying outstanding loans. When $\tilde{\delta}_2 < 1$, it means that there is better investment opportunity rather than the one that bank has now.

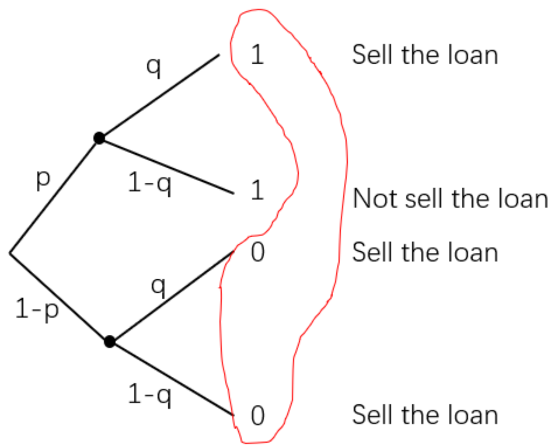


FIGURE 18. Pay-off Structure with Secondary Market

4.2.2. *Secondary Market.* If the market is liquid, then investors believe that the bank is selling loans either because the project failed (which occurs with probability $1-p$) or that the project succeeded but the bank received an attractive outside opportunity (which occurs with probability pq). Thus, if the market is liquid, outside investors value one promised date 2 dollar at a price r, where

$$r = \frac{pq}{1-p+pq} \tag{111}$$

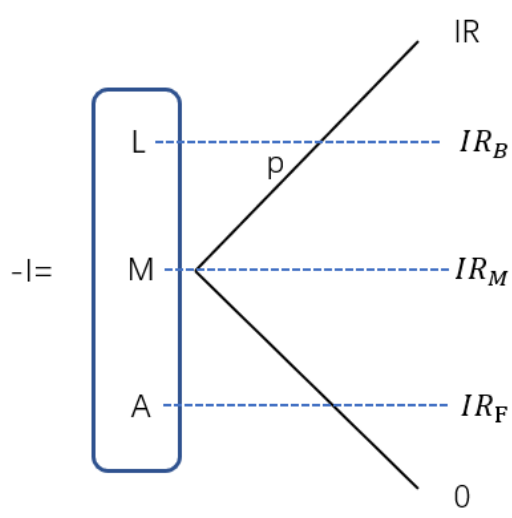


FIGURE 19. Pay-off Structure with Secondary Market

4.2.3. *Scenario I: Without Secondary Market.* For a given investment size I, the firm gives $R_B I$ final payoff to the bank, to secure a loan of size L, and $R_M I$ final payoff to the outside investors, to secure a bond of size M. The firm's problem is:

$$\begin{aligned} \max_{I,L,M,R_F,R_B} & pR_F I - A \\ pR_F - 0R_F \geq b_F & \qquad (IC_F) \\ \delta_1(E\delta_2)pR_B - 0 \geq B_B & \qquad (IC_B) \\ \delta_1(E\delta_2)pR_B I \geq L & \qquad (IR_B) \\ p(R - R_F - R_B)I \geq M & \qquad (IR_M) \\ I \leq L + M + A & \end{aligned} \tag{112}$$

In equilibrium, bank will monitor the firm and firm, if monitored, will not shirk (IC constraints).

4.2.4. *Scenario II: With secondary market.* When there is a liquid secondary market, the market price of claims at $t = 1$ is r.
At $t = 1$, the claim have a probability of $(pq + 1 - q)$ to worth r and a probability of $p(1 - q)$ to worth 1, so the value of claim for bank at $t = 1$ is $(pq + 1 - q) * \frac{pq}{1-p+pq} + 1 * p(1 - q) = p$.
The firm's problem is:

$$\begin{aligned} \max_{I,L,M,R_F,R_B} & pR_F I - A \\ pR_F - 0R_F \geq b_F & \qquad (IC_F) \\ \delta_1 pR_B \geq B_B + \delta_1 rR_B & \qquad (IC_B) \\ \delta_1 pR_B I \geq L & \qquad (IR_B) \\ p(R - R_F - R_B)I \geq M & \qquad (IR_M) \\ I \leq L + M + A & \end{aligned} \tag{113}$$

Note that the IC condition and IR condition of bank change. In IC condition, bank monitors and get return at $t = 2$, which is better than no monitoring and sell the loan with price r at $t = 1$. Bank does not have stochastic discount factor with secondary market because bank can always sell the loan and invest in new project. Same for bank IR condition.
Compare the new constraints with the old ones.
IR condition without secondary market:

$$\delta_1(E\delta_2)pR_B I \geq L \qquad (IR_B) \tag{114}$$

IR condition with secondary market:

$$\delta_1 pR_B I \geq L \qquad (IR_B) \tag{115}$$

We can find that the IR condition for bank is easier to meet with secondary market because the cost of bank capital is lower: The price of the loan no longer features the liquidity premium $E\delta_2$.
IC condition without secondary market:

$$\delta_1(E\delta_2)pR_B - 0 \geq B_B \qquad (IC_B) \tag{116}$$

IC condition with secondary market:

$$\delta_1 pR_B \geq B_B + \delta_1 rR_B \qquad (IC_B) \tag{117}$$

With secondary market, bank does not require a liquidity premium on date 2 cash flow and this will increase the incentive to monitor. But bank can choose not to monitor and sell the loan at market price, this will decrease the incentive to monitor.

5. Bank Run I

5.1. *** Diamond and Dybvig (1983, JPE).** This paper studies the role of banks in providing liquidity.

5.1.1. *Settings.*
Asset (Tree). There are three periods. At $t = 0$, invest 1 unit of seed. At $t = 2$, the tree will be worth $R = 2$. If we cut the tree at $t = 1$, we can only get 1.

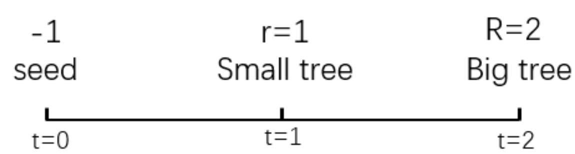


Figure 20. Timeline

Investor. The investor need to consume at $t = 1$ or $t = 2$, but he doesn't know at which date when at date 0. At $t = 1$, there is a 25% chance of having a liquidity shock. And the liquidity shock is a private information that cannot be verified. The investor's expected utility is given by

$$\pi u(c_1) + (1 - \pi)u(c_2) \tag{118}$$

where $c_1 = r = 1$ and $c_2 = R = 2$. Utility function $u(c) = 1 - \frac{1}{c}$ is a concave function.

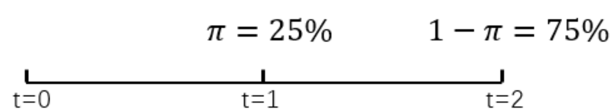


Figure 21. Stochastic Liquidity Demand

5.1.2. *Scenario I: Autarky (no trade).* The expected utility for investor in an autarky economy is

$$Eu(X^A) = \frac{1}{4}u(1) + \frac{3}{4}u(2) = 0.375 \tag{119}$$

Suppose the investor can buy insurance that gives a payoff $X^I = (1.28, 1.813)$. The expected utility of insurance for investor is

$$Eu(X^I) = \frac{1}{4}u(1.28) + \frac{3}{4}u(1.813) = 0.391 > 0.375 \tag{120}$$

The risk-averse investors are willing to give up some expected return to get a more liquid asset.
But this insurance is not available because liquidity shock is private information.

Bank. Bank can provide the more liquid asset by offering demand deposit. Suppose the bank receives \$1 from each of 100 investors, it receives \$100 in deposits on $t = 0$. The bank offers to pay 1.28 to those who withdraw at $t = 1$, thus for the investor who withdraw at $t = 2$

$$\frac{(100 - 25 * 1.28) * 2}{75} = 1.813 \tag{121}$$

25 investors will withdraw 1.28 at $t = 1$, leave $(100-25*1.28)$ in bank and the return is 2 at $t = 2$. The rest 75 investors will receive 1.813 each. IC condition for the investors is $c_2 > c_1$ so that they will not withdraw the money early when they do not have liquidity shock.

Social Planner. Suppose the payoff of optimal insurance is $X^{I^*} = (r_1, r_2)$. The social planner will maximize the expected utility of investor:

$$\begin{aligned} & \max_{r_1, r_2} \pi u(r_1) + (1-\pi)u(r_2) \\ s.t. \qquad & r_2 = \frac{(1 - \pi r_1) * R}{1 - \pi} \\ & r_2 > r_1 \end{aligned} \tag{122}$$

Solve the maximization problem and we get $(r_1, r_2) = (1.28, 1.813)$.

Fragility and Bank Run. Suppose n investors want to withdraw at $t = 1$ simultaneously. Then the payoff at $t = 2$ will be

$$r_2 = \frac{(100 - n * 1.28)2}{100 - n} \tag{123}$$

- If n=25, the payoff will be (1.28, 1.813).This is a good self-fulfilling prophecy.
- If n increase, r_2 will decrease (*erosion*).
- If $n > 56.25$, then $r_2 \leq 1.28$.
- If n=100, then payoff will be (1,0). This is a self-fulfilling prophecy: everyone expect the bank to fail and they rush to withdraw their deposit. And the sudden withdrawals lead bank to fail.

Policy.

- Suspension of convertibility: The bank does not allow more than a fraction π of deposits to be withdrawn. As a result, the depositors would never panic and a run would never start.
- Deposit Insurance: This is a promise to pay the amount promised by the bank no matter how many depositors withdraw. In this case the government makes announcement that it will support the bank. Then people change their beliefs and only π proportion of investor will withdraw at $t = 1$.

5.1.3. *Scenario II: Secondary Market.* Suppose there are two technologies now.

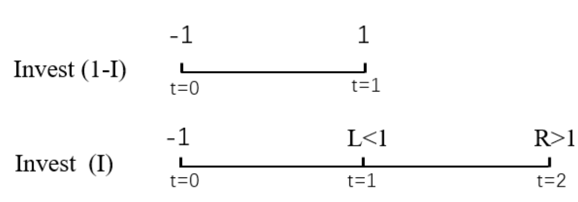


Figure 22. Two Technologies

Autarky. If there is no trade and there is still a 25% chance of having a liquidity shock for investor. The maximizing problem for investor is:

$$\begin{aligned} &\max_I \pi u(c_1) + (1-\pi)u(c_2) \\ s.t. \quad & c_1 = (1-I)*1 + I*L < 1 \\ & c_2 > (1-I)*1 + I*R < R \end{aligned} \tag{124}$$

Secondary Market. Suppose that investor can sell tree 1 at price \$1 or tree 2 at price P at $t = 1$. The expected utility is

$$\begin{aligned} &\max_I \pi u(c_1) + (1-\pi)u(c_2) \\ s.t. \quad & c_1 = (1-I)*1 + I*P \\ & c_2 = \frac{(1-I)}{P}*R + I*R \end{aligned} \tag{125}$$

- Investor facing liquidity shock at $t = 1$: sell tree 2 at price P and buy tree 1.
- Investor not facing liquidity shock: consume at $t = 2$, sell tree 1 at price \$1 and buy tree 2 with price P.
- Second constraint is a function of the first one: $c_2 = \frac{c_1}{P}R$.

Equilibrium: $P = 1$ and consumption is $(c_1, c_2) = (1, R)$.

- Suppose $P > 1$
  - type 1 investor: consumption $(1-I)*1 + I*P = 1 + I(P-1)$. Investment I will increase to 1 and type 1 investor will only choose tree 2 at $t = 0$.
  - type 2 investor: consumption $\frac{(1-I)}{P}*R + I*R = \frac{R}{P} + \frac{(P-1)IR}{P}$. Investment I will increase to 1 and type 1 investor will only choose tree 2 at $t = 0$.
  - nobody will choose tree 1 at $t = 0$, thus there will be no market.
- Suppose $P < 1$
  - Opposite to the first case, nobody will choose tree 2 at $t = 0$ and there will be no market.

At time $t = 1$, the market of tree 2:

- Demand: type 2 investor$(1-\pi)$ will buy tree 2. They use the money from selling tree 1:$(1-\pi)(1-I)$
- Supply: type 1 investor(π) will sell tree 2. They have (πI) amount of tree 2
- Price of tree 2: $P = \frac{(1-\pi)(1-I)}{\pi I} = 1$. Thus $1 - \pi = I$. The proportion of long-term investor is the same as the amount invested in tree 2(long-term investment).

Social Planner (second-best). The expected utility is

$$\begin{aligned} &\max_I \pi u(c_1) + (1-\pi)u(c_2) \\ s.t. \quad & \pi c_1 = 1 - I \\ & (1-\pi)c_2 = I*R \end{aligned} \tag{126}$$

In this case, the social planner can invest more in the short-term investment and $c_1^* > 1$, $c_2^* < R$.

5.1.4. *Comparison.*

- *autarky*: the consumption is $(c_1 < 1, c_2 < R)$;
- *ex post*: (market): the consumption is $(1, R)$;
- *ex ante*: (social planner): the consumption is $(c_1^* > 1, c_2^* < R)$;

5.2. * Two Extensions on Diamond and Dybvig (1983).

5.2.1. *Extension 1: Debt vs. Equity Contract.* In what follows I study a variant of Diamond and Dvbvig (1983), where the financial intermediate is a mutual fund rather than a bank.

5.2.2. *Set-up.* Consider the following problem. The utility of depositors is given by

$$u(c_1, c_2) = \pi u(c_1) + (1-\pi)u(c_2)$$

where π is the probability for the agent to be the early type of consumer (consumed in period 1). There are two investment technology, one is long term investment which yields return $R > 1$. There is also a short term investment technology, which transfers one dollar investment to 1 dollars across time. Each unit long-term investment yields $\lambda < 1$ unit return if it is liquidated in period 1 . Each agent is endowed with 1 dollar. We assume that $-cu''(c)/u'(c) > 1$
Mutual Fund. Now consider there is is mutual fund, who issue stocks to consumer at price of 1 in period 0. It invest d in the short term investment and $1-d$ in long term investment. It issues dividend d and $R(1-d)$ to the share holders in period 1 and period 2, respectively. There is also a stock market in period 1, in which the early consumer and later consumer can trade the mutual funds' stock at at price p, so the we have

$$\begin{aligned} c_1 &= d + p \\ c_2 &= (1 + \frac{d}{p})R(1-d) \end{aligned}$$

Asset Price. At date 1, the early type sell stock to the later consumers, in return for dividends claimed by the later. The price of mutual fund's stock can be pinned down by the following condition:

$$\pi p = (1-\pi)d$$

or

$$p = \frac{(1-\pi)}{\pi}d$$

We can show that mutual fund can achieve the first best allocation by carefully choosing d. Recall the following condition must be satisfied from first-best allocation:

$$u'(c_1^*) = Ru'(c_2^*) \tag{127}$$

with the condition:

$$c_1 = (1 + \frac{1-\pi}{\pi})d = \frac{1}{\pi}d \tag{128}$$

and

$$c_2 = (1 + \frac{(1-\pi)}{\pi})R(1-d) = \frac{R}{\pi}(1-d) \tag{129}$$

The first-best allocation be restored by setting d as d^*:

$$u'(\frac{1}{\pi}d^*) = Ru'(\frac{R}{\pi} - \frac{R}{\pi}d^*) \tag{130}$$

As LHS must be greater than RHS when d is close to zero, and RHS must be greater than LHS when d is close to one, such d^* must exist (and is unique).

The economy with this mutual fund is free from bank run equilibrium. By design the more people choose to withdraw at first period, the benefit from being patient, i.e. buy stock from them and consume later, is higher as stock price drops. The key difference is that the total dividend early consumer can claim is fixed, so bank-run will only induce redistribution within earlier consumers, leaving late consumer no worse off.

5.2.3. *Extension 2: Inter-bank Asset Market and Multiple Equilibria.* In what follows I study a variant of Diamond and Dvbvig (1983). I maintain the assumption that depositors are risk-neutral and *ex ante* identical, and random realization of liquidity shock at interim period. Therefore, depositors with realized discount factor lower than certain cut-off choose not to roll over. In this extension model we dispersion in preference shock approaching 0. Deposit from creditors constitutes asset of banks (F=1), which are held in long-term project with return R_i at date 2, or endogenous liquidation value l_i at date 1. The contract between depositor h and bank i promises a payment of 1 if depositor withdraws at date 1, and of R if depositor rolls over to date 2. We allow for partial liquidation.
Homogeneous Bank. We start with scenario where all banks invest in the same long-term project so that $R_i = R$ for all bank i. At date 1, each creditor h solves the following problem given realized liquidity shock β_h

$$V_h(\beta_h) = \max\{c_1, \beta_h c_2\} \tag{131}$$

c_1 is the amount creditors collect if call, and c_2 is the claim if roll over. We denote liquidation price as $l_i = l(\varphi)$, where φ is fraction of asset the bank liquidates. We assume β_h follows a distribution $\Phi(\beta)$ with support on [1/R-e,1/R+e], where $e \to 0$. Denote cut-off β, if exists, the associated mass of calling creditors and asset to liquidate as $\hat{\beta}$, $\hat{\lambda}$ and $\hat{\varphi}$, therefore the creditor with cut-off $\hat{\beta}$ must be indifferent between calling and holding the debt:

$$\min\{1, \frac{\hat{\varphi} l(\hat{\varphi})}{\hat{\lambda}}\} = \hat{\beta} \min\{\frac{(1-\hat{\varphi})R}{1-\hat{\lambda}}, R\} \tag{132}$$

which implies if $\hat{\varphi} < 1$

$$1 = \hat{\beta} \min\{\frac{(1-\hat{\varphi})R}{1-\hat{\varphi} l(\hat{\varphi})}, R\} = \begin{cases} \hat{\beta}R, & if \quad l(\hat{\varphi}) \geq 1 \\ \hat{\beta}R\frac{(1-\hat{\varphi})}{1-\hat{\varphi}l(\hat{\varphi})}, & if \quad l(\hat{\varphi}) < 1 \end{cases} \tag{133}$$

and if $\hat{\varphi} = 1$ (in which case $\hat{\lambda} = 1$ and $\hat{\beta} = 1$ must hold in equilibrium)

$$\frac{l(1)}{1} = R - 1/k \geq R \min\{\lim_{\hat{\varphi} \to 1} \frac{(1-\hat{\varphi})}{1-\hat{\varphi} l(\hat{\varphi})}, 1\} \tag{134}$$

5.2.4. *Scenario 1: constant liquidation price and only one bank.* We start by consider an asset market with constant liquidation price, $l(\varphi) = l$. With $\sigma(\beta)$ approaching 0, the equilibrium is unique.

Case 1: $l \geq 1$

If the liquidation price of long-term project is no lower than 1 (i.e. early withdrawal doesn't erode asset belonging to late with-drawer), from equation 133 the equilibrium is unique: creditors with $\beta_h \geq \frac{1}{R}$ roll over while those with $\beta_h < \frac{1}{R}$ call. We denote this cut-off liquidity shock as $\beta^* = \frac{1}{R}$, cut-off mass as $\lambda^* = \Phi(\beta^*)$ and cut-off liquidated asset as φ^* such that $\pi(\varphi^*) = \lambda^*$.

Case 2: $l < 1$

If the liquidation price of long-term project is lower than 1, from equation 134 the bank-run equilibrium always exists: all creditors call at date 1 and receive $l < 1$. It can be shown that when dispersion of liquidity shock β is sufficiently close to 0, the bank-run equilibrium becomes the only equilibrium left. In this case given $l < 1$, *ex ante* no creditor would save in bank at date 0.

5.2.5. *Scenario 2: risk averse investors and only one bank.* When investors at asset market are risk-averse , i.e. market depth is finite, the liquidation price is a decreasing function of liquidated asset. With one bank, we allow for partial liquidation. We denote liquidation price as $l_i = l(\varphi)$, where φ is fraction of asset the bank liquidates. We adopt the functional form in the paper: $l(\varphi) = R - \varphi/k$, where k measures market depth[15].

Case 1: *(k is high)* $k > \frac{1}{R-1} \Rightarrow l(1) \geq 1$

If investors are *almost* risk-neutral, i.e. market is deep enough, the liquidation price of long-term project is high than 1 even if all creditors choose not to roll over. This corresponds to case 1 in the first scenario that the equilibrium is unique (from equation 133), that creditors with $\beta_h \geq \frac{1}{R}$ roll over while those with $\beta_h < \frac{1}{R}$ call. This is the equilibrium E^* in Panel A of *Figure 1* below.

Case 2 *(k is intermediate)* $\frac{1}{R-1} > k > \frac{\Phi(\frac{1}{R})}{R-1} \Rightarrow l(0) > l(\lambda^*) \geq 1 > l(1)$

Firstly, the first best equilibrium above can be sustained [16] In this equilibrium, the cut-off β is again $\beta^* = \frac{1}{R}$, such that creditors with severe liquidity shock ($\beta_h < \frac{1}{R}$) call to collect 1 unit at date 1, and creditors with $\beta_h > \frac{1}{R}$ roll over to claim R units at date 2.

Secondly, there exists one unstable equilibrium with $\tilde{\beta} > \beta^*$ [17]. These two are equilibrium E1 and E2 in Panel B of *Figure 1*.

Thirdly, there exists a stable bank-run equilibrium (from equation 134).

Case 3: *(k is low)* $\frac{\Phi(\frac{1}{R})}{R-1} > k > \frac{2}{R}$ so that $l(0) > 1 > l(\lambda^*) > l(1)$

There always exists a bank-run equilibrium (see Panel C of *Figure 1*). Similar to discussion on case 2 of first scenario, with $\sigma(\beta)$ approaching 0 LHS is always greater than RHS of equation 133, such that creditors always choose to call on the debt and get no more than 1. Then at date 0, as market depth is limited, *ex ante* no creditor is willing to invest in bank.

5.2.6. *Insight.* When liquidation price curve is downward-sloping and $\sigma(\beta)$ approaches 0, multiple equilibria arise when market depth (parameter k in the model) is intermediate. If k further increases or decreases, the equilibrium becomes unique as in the paper. Therefore, whether the multiple equilibria exist does depend on fundamental (μ_θ or k here) even in this Diamond-Dybvig (1983) framework.

There are two levels of *strategic complementarity* and their *two-way feedback* loops:

1. level one: strategic complementarity between depositors on roll-over decision that affects liquidation decision of bank managers
2. level two: strategic complementarity between bank managers on liquidation decision that affects liquidation price of assets
3. feedback one: lower liquidation price $\rightarrow$ lower return to roll-over $\rightarrow$ fewer roll-over

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[15]Liquidation value is $\pi(\varphi) = \varphi l(\varphi)$. We set $k > \frac{2}{R}$ to guarantee that $\pi'(\varphi) > 0$ over the domain of φ.

[16]We verify this by showing that if only $\lambda^* = \Phi(\frac{1}{R})$ fraction of creditors liquidate, the banker only needs to liquidate an amount that is less than λ^* of total asset: $\pi(\lambda^*) = \lambda^* l(\lambda^*) \geq \lambda^*$. In other word, early liquidation doesn't erode asset of late type. A sufficient condition for last equation is $l(\lambda^*) \geq 1$, or $k > \frac{\lambda^*}{R-1} = \frac{\Phi(\frac{1}{R})}{R-1}$.

[17]*Sketch of proof:* Denote the cut-off shock, mass of calling creditors and liquidated asset in unstable equilibrium as $\tilde{\beta}$, $\tilde{\lambda}$ and $\tilde{\varphi}$.

$$1 = \begin{cases} \hat{\beta}R, & if \quad l(\varphi) \geq 1 \\ \hat{\beta}R\frac{(1-\varphi)}{1-\varphi l(\varphi)}, & if \quad l(\varphi) < 1 \end{cases} \tag{135}$$

In equation above $RHS > LHS = 1$ if $\varphi = \varphi^* + \varepsilon$; $LHS > RHS = 0$ if φ approaches 1. By continuity there must exist a cut-off $1 > \tilde{\varphi} > \varphi^*$.

depositors
4. feedback two: fewer roll-over depositors → more liquidation → lower liquidation price

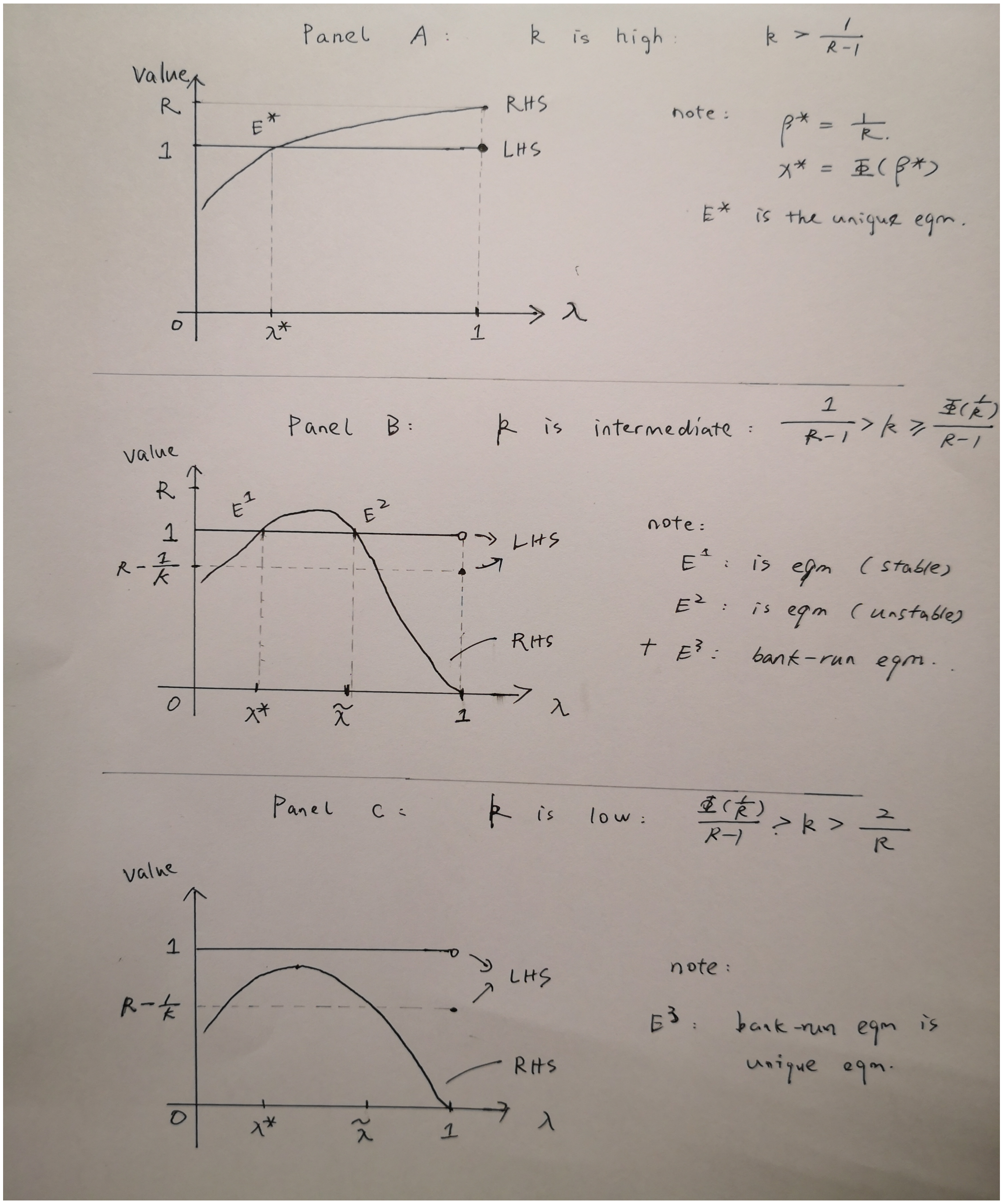


FIGURE 23. Existence of Multiple Equilibria with Inter-bank Asset Market

6. Bank Run II: Global Games

6.1. *** Morris and Shin (1998, AER).** This paper popularizes the global-games approach[18] and studies an application to currency attack/bank run.

Settings. Two players are deciding whether to invest (or attack, run etc). The payoff structure is given as below:

	Invest	NotInvest
Invest	θ, θ	$\theta - 1, 0$
NotInvest	$0, \theta - 1$	$0, 0$

Figure 24. Payoff Structure

Complete Information. If there is complete information about θ, there would be three cases to consider:
1) If $\theta > 1$, each player has a dominant strategy to invest.
2) If $\theta \in [0, 1]$, there are two pure strategy Nash equilibria: both invest and both not invest.
3) If $\theta < 0$, each player has a dominant strategy not to invest.

Incomplete Information. Player i observes a private signal $x_i = \theta + \varepsilon_i$ with $\varepsilon \sim N(0, \sigma^2)$. The player observed signal x considers θ to be normally distributed with mean x and standard deviation σ: $\theta | x \sim N(x, \sigma^2)$. When σ is *exactly* zero, the model reduces to the complete-information benchmark.
When, instead, σ is positive but small enough, every agent is nearly perfectly informed, and we have a seemingly tiny perturbation in the exogenous primitives of the environment. One may have expected that such a tiny change in the assumptions of the model would imply a tiny difference in its predictions. This turns out not to be the case. The predictions of the theory are discontinuous at $\sigma = 0$. To see this, since $x_i - \varepsilon_i = x_j - \varepsilon_j$, we have $x_j = x_i - \varepsilon_i + \varepsilon_j$. Then player i thinks his opponent's signal x is $x_j | (x_i = x) \sim N(x, 2\sigma^2)$.
Now let's try to understand the discontinuity at $\sigma = 0$, i.e. sharp difference between complete and incomplete information. The following section is quoted from handbook chapter by Angeletos and Lian (2016) on *fundamental uncertainty* vs. *strategic uncertainty*:

"First, note that equilibrium imposes that agents know one another's strategies, that is, they know the mappings from their information sets (or Harsanyi types) to their actions. If we assume that all agents share the same information, then this imposes that all agents face no uncertainty about their actions. The absence of this kind of strategic uncertainty is conducive to multiple equilibria: it is "easy" to coordinate on one of many equilibrium actions when the agents are confident that other agents will do the same. But once information is incomplete, the agents may face uncertainty about one another's actions, and this type of uncertainty can hinder coordination. It follows that the determinacy of the equilibrium hinges on the level of strategic uncertainty: the higher the level of strategic uncertainty, the harder to sustain multiple equilibria.

[18] Global games are games of incomplete information where players receive possibly-correlated signals of the underlying state of the world.

"Next, note that the level of strategic uncertainty is not necessarily tied to the level of fundamental uncertainty: the uncertainty an agent faces about the beliefs and actions of other agents has to do more with the heterogeneity of the information and the associated higher-order uncertainty, and less with the overall level of noise in the observation of the fundamentals. In fact, when private information becomes more precise, the uncertainty that an agent i faces about the fundamentals necessarily decreases, yet it is possible that her uncertainty about beliefs and actions of any other agent j increases. This is because an increase in the precision of private information implies that the beliefs and actions of agent j become more anchored to her own private information, which is itself unknown to agent i. This anchoring effect in turn explains why private information can exacerbate higher-order uncertainty and thereby hinder coordination.

"This intuition can be formalized as follows. First, note that, when information is complete, the equilibrium belief of any agent about x is a direct measure of the realized value of x: That is, agents are perfectly informed about the size of the attack in equilibrium, irrespective of the equilibrium selected. Next, note that, in the diametrically opposite scenario where an agent is completely agnostic about the size of the attack, her belief about x is uniform over the [0; 1] interval. Finally, consider what happens under incomplete information. Let σ be small enough so that the equilibrium is unique and consider the "marginal" agent, that is, the type who is indifferent between attacking and not attacking in equilibrium. "

Equilibrium Strategy. Without any loss, we will assume that an agent attacks whenever she is indifferent between attacking and not attacking. A nature kind of strategy is the player will invest only if he observes a private signal above some cutoff point x^*.

$$S(x) = \begin{cases} \text{invest} & if \quad x_i > x^* \\ \text{not invest} & if \quad x_i \leq x^* \end{cases}$$

Given x, player i assign probability $Pr(x_j < x^*) = \Phi((x^* - x)/\sqrt{2}\sigma)$ to his opponent j observing a signal less than x^*.
So player i's expected utility given his opponent' probability would be:

$$\begin{aligned} E[(1 - \Phi(\frac{x^* - x}{\sqrt{2}\sigma}))\theta + \Phi(\frac{x^* - x}{\sqrt{2}\sigma})(\theta - 1)|x_i] \geq 0 \\ E[\theta - \Phi(\frac{x^* - \theta}{\sqrt{2}\sigma})|x_i] \geq 0 \\ x - \Phi(\frac{x^* - x}{\sqrt{2}\sigma})] \geq 0 \end{aligned} \tag{136}$$

When $x^* = 1/2$, the equality holds. the unique equilibrium has both players investing only if they observe a signal greater than 1/2.

Message. Although multiple equilibria exist when the fundamentals (such as reserves) are common knowledge, a unique equilibrium is obtained by adding a small amount of idiosyncratic noise in the speculators' information about the fundamentals. The size of the attack and the devaluation outcome no longer depend on sunspots but may exhibit a strong nonlinearity (or near-discontinuity) with respect to the fundamentals.

6.2. *** Liu (2021).** This paper studies bank runs in a financial system, featuring the interaction between systemic bank runs and endogenous liquidation prices in a dynamic environment. The paper features two levels of *strategic complementarity* and their *two-way feedback* loops:
1. level one: strategic complementarity between depositors on roll-over decision that affects liquidation decision of bank managers
2. level two: strategic complementarity between bank managers on liquidation decision that affects liquidation price of assets
3. feedback one: lower liquidation price → lower return to roll-over → fewer roll-over depositors
4. feedback two: fewer roll-over depositors → more liquidation → lower liquidation price

Set Up.
Timeline. three-date static model

- date 0: bank i invests in one unit of its own assets at the cost of 1. The aggregate state of the economy is μ_θ.
  - F comes from a continuum of its creditors (depositors) with F mass
  - 1 -F comes from its equity holder
- date 1: Bank's asset quality is realized: $\theta_i \sim N(\mu_\theta, \sigma_\theta^2)$. Creditor can observe a signal of asset quality: $s_i^h = \theta_i + \sigma_s \epsilon_e^h$.
  - creditors of a bank decide whether or not to roll over his lending to the bank (see pay-off below):
  - if call & bank survives: 1
  - if roll-over & bank survives: R
  - if call & bank fails: $\frac{l_i}{F}$, where l_i is endogenous liquidation value
  - if roll-over & bank fails: 0
- date 2: pay-off to bank's asset is realized $v_i \equiv \theta_i + e_i$, where $e_i = e \sim N(0, \sigma_e^2)$.

Pay-off.

	Total Calling Proportion $\lambda \in [0, \frac{l_i}{F}]$	Total Calling Proportion $\lambda \in [\frac{l_i}{F}, 1]$
Hold	$\min\{R, \frac{v_i}{F}\}$	0
Call	1	$\frac{l_i}{F}$

Endogenous liquidation price. Solving the investor's problem (skipped), Lemma 1 gives the liquidation price of bank i's asset:

$$l_i = \theta_i - \varphi/k$$

where $k = n/(\gamma\sigma_e^2)$ measures market thickness. μ denotes endogenous measure of banks suffering creditor runs. In other word, there are μ units of assets in the system under fire sales.

Equilibrium at Date 1.
Assumption 1. We consider the limiting case with $\sigma_s \to 0$, i.e. fundamental uncertainty disappears.
Assumption 2. We consider an equilibrium where every creditor uses a threshold (monotone) strategy, such that creditors receiving signal s_i^h higher than a cut-off s^* will roll over, while those with signal $s_i^h < s^*$ will call.

Denote the expected pay-off of debt at t=2, conditional on bank doesn't fail at date 1 as

$$D(\theta_i; R) = E\{\min(R, \frac{v_i}{F})|\theta_i\}$$

The equilibrium of the creditor-run game for each individual bank i is then given by:

$$\underbrace{D(s^*;R)\frac{l_i}{F}}_{\text{expected pay-off of holding}} = \underbrace{\frac{l_i}{F}+\frac{l_i}{F}(1-\frac{l_i}{F})}_{\text{expected pay-off of calling}} \tag{137}$$

where $l_i = s^* - \varphi/k$. The LHS of equation (137) is simply conditional pay-off $D(s^*;R)$ times probability of survival $\frac{l_i}{F}$. If the agent calls, with probability $\frac{l_i}{F}$ bank survives and he receives 1, and with probability $1-\frac{l_i}{F}$ bank fails and he receives $\frac{l_i}{F}$.
The intuition is as follows: In making rollover decisions, an individual creditor faces *fundamental uncertainty*(i.e. he is not sure whether asset quality is high) as well as *strategic uncertainty* (i.e., he is not sure whether peer creditors of the same bank will roll over or not). Under the limit $\sigma_s \to 0$, fundamental uncertainty disappears. Thus to the marginal creditor who receives signal s^*, his inference on asset quality is $\theta_i = s^*$, and his inference to liquidation price is $l_i = s^* - \phi/k$. However, However, strategic uncertainty does not disappear under the limit $\sigma_s \to 0$. To the marginal creditor, he perceives that λ (i.e., the proportion of peer creditors choosing to call) is uniformly distributed within [0, 1]. Hence, in his eyes, the probability that the proportion of creditors calling loans is less than $\frac{l_i}{F}$ is simply $\frac{l_i}{F}$, i.e. his perceived probability of bank failure is $\frac{l_i}{F}$.

We can rewrite last equation as:

$$D(s^*;R) - 1 = 1 - \frac{l_i}{F} \tag{138}$$

Banks with realized asset quality $\theta_i > \theta^* = s^*$ survive at t = 1 while all others fail. Thus total measure of failing bank is

$$\varphi = \Phi(\frac{s^* - \mu_\theta}{\sigma_\theta}) \tag{139}$$

Two way feedback loops. Two-way feedback exists between liquidation prices (ϕ or l_i) and the run threshold (s^*), if

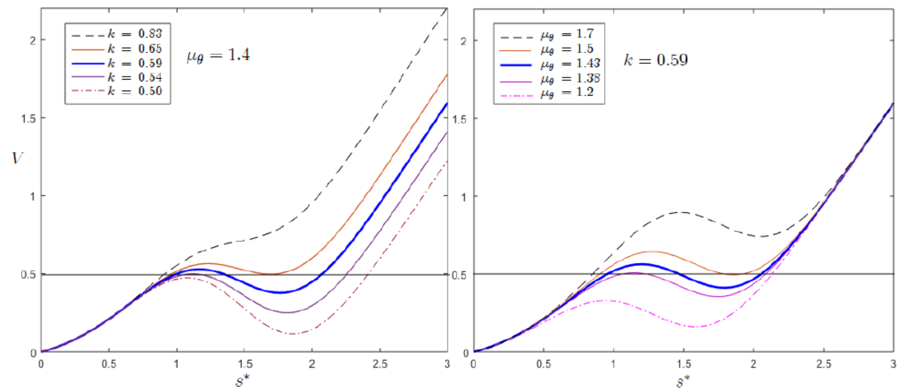
- $\frac{\partial \phi}{\partial s^*} > 0$: When creditors run on banks with a higher threshold, more banks in the system will fail, resulting in a lower liquidation price for every bank.
- $\frac{\partial s^*}{\partial \phi} > 0$: Creditors of a bank have rational expectations on this and thus have higher incentives to run in the first place.

Credit-Run market equilibrium. Combining last two equations, we have the equation that fully characterize the equilibrium

$$V(s^*,\mu_\theta) \equiv (1 - \frac{s^* - \Phi(\frac{s^*-\mu_\theta}{\sigma_\theta})/k}{F})\frac{1}{D(s^*;R)-1} = 1 \tag{140}$$

- When k is high enough or μ_θ is low enough, the equilibrium is always unique.
- For a given sufficiently high μ_θ, when k is low enough or high enough, there is a unique equilibrium. When k is in an intermediate range, there exist multiple (typically three) equilibria, where two of them are stable equilibria.
- For a given sufficiently low k, when μ_θ is low enough or high enough, the equilibrium is unique; when μ_θ is in an intermediate range, there exist multiple (typically three) equilibria where two of them are stable equilibria.

Even under the limit $\sigma_s \to 0$, multiple equilibria can exist at the system level, which is in contrast to the classical result of the bank-run game for a single bank. Intuition: The presence of a common asset market gives rise to strategic complementarities among creditors of different banks, besides the complementarities among creditors of the same bank. In fact, a necessary condition for multiple equilibria is that strategic complementarity be strong enough.



[14]In Figure 4, given $\mu_\theta = 1.4$, when $0.54 \leq k \leq 0.65$, the equation admits three solutions; when $k < 0.54$ or $k > 0.65$, there is a unique solution. Given $k = 0.59$, when $1.38 \leq \mu_\theta \leq 1.5$, the equation admits three solutions; when $\mu_\theta < 1.38$ or $\mu_\theta > 1.5$, there is a unique solution.

FIGURE 25. Equilibria in Static Model

Comment. One condition to ensure that equation (140) characterizes equilibrium:

- *Ex ante* participation constraint: creditors are willing to save at the bank at date 0 (expected return is higher than 1), so that $F > 0$. (This condition is given as equation (8) in the other *"Diversification"* paper, but absent in current work.) Moreover, when k and μ_θ is low and there is a unique equilibrium, creditor may not choose to save in banks at date 0. Thus it should be checked whether F is positive or zero.

Equilibrium at date 0. The participation condition must ensure that expected return from saving in the bank ($\equiv R_0$) is high enough. Under assumption $\sigma_s \to 0$ all creditors of the same bank take the same action at date 1 ('no partial liquidation'), so that we only consider two scenarios on the equilibrium path: *"bank surviving, creditor holding"* and *"bank failing, creditor calling'*, conditional on $F > 0$:

$$R_0 \equiv \int_{-\infty}^{s^*} \underbrace{\frac{\theta_i - \Phi(\frac{s^*-\mu_\theta}{\sigma_\theta})/k}{F}}_{\text{in case of bank failure}} d\Phi(\frac{\theta_i - \mu_\theta}{\sigma_\theta}) + \int_{s^*}^{+\infty} \underbrace{E\{\min(R, \frac{v_i}{F})|\theta_i\}}_{\text{in case of bank survival}} d\Phi(\frac{\theta_i - \mu_\theta}{\sigma_\theta}) \tag{141}$$

where s^* is the cut-off solved in date-1 problem. A natural constraint (which can be used to endogenize R) is

$$R_0 \geq 1$$

7. Pecuniary Externality and Fire Sales

7.1. *** Korinek (2018, JIE).** This paper contributes to the literature on pecuniary externality, in specific collateral pecuniary externalities, that arise when market prices such as asset prices (in their context exchange rates) show up in a binding financial constraint. It shows that collateral externalities may induce emerging market borrowers to take on too much foreign currency debt (*under-borrowing*) in contrast to *over-borrowing* in Bianchi (2011) etc.

Settings.

- Agents maximize

$$U = E\sum_{t=0}^{\infty}\beta^t U(C_{Tt}, C_{Nt}) \tag{142}$$

  s.t.

$$C_{Tt} + P_t C_{Nt} + E[m^{\omega}_{t+1}B^{\omega}_{t+1}] = Y_{Tt} + P_t Y_{Nt} + B_t \tag{143}$$

$$E[m^{\omega}_{t+1}B^{\omega}_{t+1}] + \phi[Y_{Tt} + P_t Y_{Nt}] \geq 0 \tag{144}$$

- C_{Tt}: traded goods; C_{Nt}: non-traded goods
- $U(C_T, C_N)$: strictly increasing in each element, quasi-concave and homothetic
- $\omega_t \in \Omega_t$, a state of nature, is realized and observed by all each period
- (Y_{Tt}, Y_{Nt}): stochastic endowment of domestic agents and follows a Markov process
- $E[m^{\omega}_{t+1}B^{\omega}_{t+1}]$: the total amount of finance that domestic agents save in state-contingent securities

Some remarks on setting:

- International Lenders
  - take the supply of capital from international investors as given
    Do not take a stance on what determines their pricing kernel.
  - assume: $\beta < E[m^{\omega}_{t+1}], \forall t, \omega_t$, equivalent to $\beta R < 1$
  - assume moral hazard: $E[m^{\omega}_{t+1}B^{\omega}_{t+1}] + \phi[Y_{Tt} + P_t Y_{NT}] \geq 0$:
    Limits the total amount of financial liabilities in period t to ϕ of total income
  - private liability choices exhibit externalities even if face an exogenous risky funds
- financial constraint:
  - Financing capacity of private agents depends on the real exchange rate P_t.
  - Currency depreciation reduce financing capacity, give rise to "contractionary depreciation" when binding.

F.O.C.s:

- We assign the shadow prices λ_t and μ_t to the two constraints

$$[C_{Tt}] : \lambda_t = \frac{\partial U(C_{Tt}, C_{Nt})}{\partial C_{Tt}} \tag{145}$$

$$[C_{Nt}] : P_t\lambda_t = \frac{\partial U(C_{Tt}, C_{Nt})}{\partial C_{Nt}} \tag{146}$$

$$[b^{\omega}_{t+1}] : \mu^{\omega}_{t+1}(\lambda_t - \mu_t) = \beta\lambda^{\omega}_{t+1} \tag{147}$$

- Implications:

– Given their impatience, domestic agents decumulate wealth until they reach the neighborhood of the binding constraint.
– In the ergodic equilibrium, the economy fluctuates between periods of binding constraints and periods of loose constraints in that neighborhood.

Decentralized Equilibrium.

- **Definition**: a sequence of allocations $(C_{Tt}, C_{Nt}, b^{\omega}_{t+1})$ and real exchange rates P_t that satisfy the optimization problem of domestic agents and that clear the market for non-traded goods $C_{Nt} = Y_{Nt}$ and for traded goods every period.
- **Lemma 1** The economy's real exchange rate is a strictly increasing function of the ratio C_{Tt}/Y_{Nt}, i.e.
$$P_t = P(C_{Tt}/Y_{Nt}) \tag{148}$$
with $P'(C_{Tt}/Y_{Nt}) > 0$
- Intuitively, the real exchange rate adjusts to reflect the relative scarcity of traded goods in the economy.
- **Portfolio Allocation Problem and Optimal Risk-sharing**
For any state $\omega_{t+1} \in \Omega_{t+1}$, optimal risk-sharing requires that
$$\frac{\partial U(C_{Tt}, C_{Nt})}{\partial C_{Tt}} = \frac{\beta \frac{\partial U(C_{Tt}, C_{Nt})}{\partial C_{Tt+1}}}{m^{\omega}_{t+1}} + \mu_t \tag{149}$$
  – The marginal rates of substitution between domestic agents and international investors are equated across all $\forall \omega_{t+1}$ in period $t+1$.
  – If the financial constraint in period t is loose, then $\mu_t = 0$ and domestic agents also equate their intertemporal marginal rate of substitution with that of international investors, $\beta U'^{\omega}_{Tt+1}/U'^{\omega}_{Tt} = m^{\omega}_{t+1} \forall \omega_{t+1}$.
  – If international investors were risk-neutral, domestic agents obtain perfect consumption insurance across all states of nature.
  – If insurance from international investors is costly, domestic agents choose an unsmooth consumption profile that optimally trades off risk versus return.
- **Financial Amplification**
Lemma 2: In a constrained period, a marginal increase in aggregate wealth b_t relaxes the financial constraint by
$$\frac{-dE[m^{\omega}_{t+1} B^{\omega}_{t+1}]}{dB_t} = \frac{\phi P'}{1 - \phi P'} \tag{150}$$
and raises traded consumption by
$$\frac{dC_{Tt}}{dB_t} = 1 + \frac{-dE[m^{\omega}_{t+1} B^{\omega}_{t+1}]}{dB_t} = \frac{1}{1 - \phi P'} > 1 \tag{151}$$
Intuition:
  – First, more borrowing leads to a direct one-for-one increase in consumption.
  – Secondly, it relaxes the constraint and triggers financial amplification effects: higher consumption→ $P' \uparrow \to$ allows for more borrowing $\phi P' \to$ further increase in consumption and so on.
  – *For this condition to hold?*

Planner's Equilibrium.

- The social planner maximize

$$\max_{C_{Tt},B^{\omega}_{Nt}} E\sum_{t=0}^{\infty}\beta^t U(C_{Tt},C_{Nt}) \tag{152}$$

$$s.t. C_{Tt}+E[m^{\omega}_{t+1}B^{\omega}_{t+1}] = Y_{Tt}+B_t \tag{153}$$

$$E[m^{\omega}_{t+1}B^{\omega}_{t+1}]+\phi[Y_{Tt}+P(C_{Tt}/Y_{Nt})Y_{Nt}] \geq 0 \tag{154}$$

- F.O.C

$$[C_{Tt}]:\tilde{\lambda}_t = \frac{\partial U}{\partial C_{Tt}}+\tilde{\mu}_t\phi P'(C_{Tt}/Y_{Nt}) \tag{155}$$

$$[B^{\omega}_{t+1}]: m^{\omega}_{t+1}(\tilde{\lambda}_t-\tilde{\mu}_t) = \beta\tilde{\lambda}^{\omega}_{t+1} \tag{156}$$

- *Question: How does the exchange rate P compare with the decentralized problem?*
- **Proposition 1** (Constrained Efficient Allocation). The planner implements the constrained efficient allocation in the economy by imposing non-negative taxes on the sale of state-contingent Arrow securities b^{ω}_{t+1} of

$$\tau^{\omega}_{t+1} = \phi P'(C^{\omega}_{Tt+1}/Y^{\omega}_{Nt+1})\frac{\beta\tilde{\mu}^{\omega}_{t+1}}{\frac{\partial U(C_{Tt},C_{Nt})}{\partial C_{Tt}}} \geq 0 \tag{157}$$

- The intuition: we can replicate the generalized Euler equation of the planner by substituting the tax rates τ^{ω}_{t+1} into the Euler equation of private agents under taxation.
- We observe that the planner's shadow price of being constrained will satisfy

$$\tilde{\mu}_t(1-\phi P') = \mu_t \tag{158}$$

- **Corollary 1** (Regulating Capital Flows). The optimal specific tax on a capital inflow with payoff vector $t(X^{\omega}_{t+1})$ is

$$t(X^{\omega}_{t+1}) = E[\tau^{\omega}_{t+1}X^{\omega}_{t+1}] \tag{159}$$

  We thus call τ^{ω}_{t+1} the **externality pricing kernel** of the economy. If $X^{\omega}_{t+1}<0$ then equation (159) provides the optimal subsidy on capital outflows.

A Comparison. In this section I revisit the question on relation between Korinek (2018) and Benigno et al. (2013). Both papers contribute to the literature on pecuniary externality, in specific collateral pecuniary externalities, that arise when market prices such as asset prices (in their context exchange rates) show up in a binding financial constraint. Both papers show that collateral externalities may induce emerging market borrowers to take on too much foreign currency debt (*under-borrowing*) in contrast to *over-borrowing* in Bianchi (2011) etc., but through different mechanism owing to distinctive environments. Two papers differ in the following dimensions: (1) endowment economy in Korinek (2018) vs. production economy in Benigno et al. (2013). As a result, the scope of macro-prudential intervention can be extended from *ex ante* preventive policies in Korinek (2018), to *ex post* crisis mitigation policies in Benigno et al. (2013). (2) availability of state-contingent securities: which is available in Korinek (2018), while in environments of Benigno et al. (2013) foreign currency debt is the only financial contract available. (3) occasionally binding constraint: In Korinek (2018) the borrowing constraint is always binding by assumption of discount factor, while in Benigno et al. (2013) the borrowing constraint is occasionally binding.

In what follows I highlight key mechanism in two papers, and explain the intuition behind distinctive macro-prudential policies in endowment and production economy.

Endowment Economy and *ex ante* Policy in Korinek (2018). The borrowing constraint in decentralized endowment economy of Korinek (2018) is characterized by the following equation:

$$E\left[m^{\omega}_{t+1}B^{\omega}_{t+1}\right] \geq -\phi\left[Y_{Tt} + P_t Y_{Nt}\right] \tag{160}$$

where P_t denotes real exchange rate measured as relative price of non-traded goods ($C_N = Y_N$) in terms of traded goods (C_T), and is a strictly increasing function of ratio of traded to non-traded good such that

$$P_t \equiv P(C_{Tt}/Y_{Nt}), \qquad \text{and} \qquad P'(C_{Tt}/Y_{Nt}) > 0$$

However, the effect of borrowing and thus traded consumption on real exchange rate is internalized by individual borrower in decentralized equilibrium (CE), who take P_t in borrowing constraint as given. In constrained social planner (SP) equilibrium of Korinek (2018) economy, emerging market borrowers are subject to a borrowing constraint in the following form

$$E\left[m^{\omega}_{t+1}B^{\omega}_{t+1}\right] \geq -\phi\left[Y_{Tt} + P\left(C_{Tt}/Y_{Nt}\right)Y_{Nt}\right] \tag{161}$$

Policy. Therefore, constrained social planner has incentive for preventive policy to discourage saving and increase traded consumption in order to relax borrowing constraint. In Korinek (2018), the SP equilibrium can be implemented with a tax on sale of state-contingent Arrow securities B^{ω}_{t+1}, which raises the cost of saving.

Production Economy and *ex ante* & *ex post* Policy in Benigno et al. (2013). The borrowing constraint in decentralized economy of Benigno et al. (2013) takes a similar form as that of Korinek (2018):

$$B_{t+1} \geq -\phi\left[Y_{Tt} + P_t Y_{Nt}\right], \tag{162}$$

except that now Y_{Tt} and Y_{Nt} are endogenously determined by production function

$$\begin{aligned} Y_{Nt} &= A^N_t H^{1-\alpha^N}_{Nt} \\ Y_{Tt} &= A^T_t H^{1-\alpha^T}_{Tt} \end{aligned} \tag{163}$$

where H_{Nt} and H_{Tt} denote labor employed in non-traded and traded goods sector respectively, such that total labor supply is constant

$$H_{Nt} + H_{Tt} = H$$

The distinctive feature of this two-sector production economy is an interaction between production, labor and borrowing decisions: (1) As in Korinek (2018), lower C_{Tt} due to a possibly binding borrowing constraint in the future implies a lower P_t. (2) The endogenous labor allocation imposes an additional effect: for given total labor supply, the initial decline in P_t induces a shift of labor toward the traded goods sector, and hence a fall in production and consumption of non-traded goods as suggested in labor supply equations

$$\begin{aligned} W_t &= \left(1-\alpha^N\right)P_t A^N_t \left(H_{Nt}\right)^{-\alpha^N} \\ W_t &= \left(1-\alpha^T\right)A^T_t \left(H_{Tt}\right)^{-\alpha^T} \end{aligned} \tag{164}$$

If goods are complements as in the model, the ensuing decline in non-traded consumption induces agents to consume even less traded goods and to save even more compared to the endowment economy. Thus production economy can generates stronger precautionary saving than in an endowment economy.

Constrained social planner, on the other hand, not only internalizes the effect of traded consumption on value of collateral through real exchange rate, but also endogenous allocation of labor across two sectors in the borrowing constraint:

$$B_{t+1} \geq -\phi \left[A_t^T \left(H_t^T\right)^{1-\alpha^T} + \frac{(1-\omega)^{\frac{1}{\kappa}}}{\omega^{\frac{1}{\kappa}} \left(C_t^T\right)^{-\frac{1}{k}}} \left(A_t^N \left(H_t^N\right)^{1-\alpha^N} \right)^{1-\frac{1}{\kappa}} \right], \tag{165}$$

The first effect implies that, in addition to the effect in Korinek (2018) with binding constraint, when borrowing constraint is currently not binding (thus increasing traded consumption doesn't relax current borrowing constraint), raising saving may reduce the possibility of binding in the future. The second difference implies that the behavior of the economy in crisis states in the SP and the CE allocation differs (unlike endowment economy). If the social planner can alleviate the crisis compared to what happens in the decentralized equilibrium, this effect would tend to decrease the marginal value of saving in SP compared to the CE. Therefore, whether CE exerts over-borrowing or under-borrowing depends on the relative strength of these two forces.

Policy. The paper with production highlights a key rational behind optimal macroprudential policy: *ex ante* policy depends on characteristics and effects of *ex post* policy (for example, reallocate labor across sectors in the crisis). In the model, if *ex post* policies that mitigate the severity of a crisis reduce the social value of precautionary saving in normal times, ex-ante policies should be designed to induce more borrowing by private agents.

7.2. *** Davila and Korinek (2018, REStud).** This paper provides a general framework on pecuniary externalities in economies with financial frictions.

Introduction. Pecuniary externalities

- concept
  - actions of an economic agent → market price (*pecuniary*)
  - example: your action of buying a house drives up housing price
  - channel: price (not resources)
- welfare implication
  - complete market: irrelevant (Pareto efficient)
    if someone buys a house and pushes up housing price, the buyer of houses will be worse off and the sellers will be better off. Importantly, the loss to consumers is precisely offset by the gain to producers. If market is complete, the allocation would be Pareto efficient.
  - incomplete market: relevant (MU/MP of agents $\neq$).
- particularly true for financial economics
  - financial constraint: pecuniary externalities ⇒ $\Delta price$ ⇒ first-order welfare implications
  - phenomena: fire sales and financial amplification etc.
  - justification for macro-prudential regulation
- two types of pecuniary externalities
  - distributive externalities
    * zero-sum in any given state;
    * relevant when MRS between states $\neq$ among agents;
    * example: fire sales and terms of transaction
  - collateral externalities
    * asset price → financial constraint
    * example: fire sales and financial constraint
- sign of pecuniary externalities:
  - distributive externalities: 3 sufficient statistics
    * difference in MRS of agents
    * trading position of capital and financial assets
    * sensitivity of equilibrium price to changes in sectoral state variables
  - collateral externalities: 3 sufficient statistics
    * shadow value of binding financial constraint (+)
    * sensitivity of financial constraint to asset price (+)
    * sensitivity of equilibrium price to changes in sectoral state variables

Baseline Model

A Canonical Model of Kiyotaki and Moore:

- two agents $i \in I$ of measure 1
  - borrower (b): productive, financially constrained etc.
  - lender (l)
- two goods
  - consumption good ("numeraire")
  - capital good
- three periods with uncertainty on aggregate state $\omega \in \Omega$
  - date 0

  - date 1 (ω)
  - date 2 (ω)
- preference: time separable utility function

$$U = E_0[\sum_{t=0}^{2} \beta^t u(c_t)] \tag{166}$$

Timeline

- date 0
  - receive endowment e_0
  - consumption: c_0
  - investment: $h(k_1) \rightarrow k_1$
  - contingent security: $E_0[m_1^\omega x_1^\omega]$
- date 1 (ω realized)
  - receive $e_1^\omega + x_1^\omega + F_1^\omega(k_1)$ consumption good
  - consumption: c_1^ω
  - buy/sell capital good: $q^\omega \Delta k = q^\omega(k_2 - k_1) \rightarrow$ date 2: k_2
  - bond: $m_2^\omega x_2^\omega$ (m_2^ω: market discount factor)
- date 2
  - consume $e_2^\omega + x_2^\omega + F_2^\omega(k_2)$ consumption good
- budget constraint

$$\begin{aligned} e_0^i &= c_0^i + h^i(k_1^i) + E_0[m_1^\omega x_1^{i,\omega}] \\ e_1^{i,\omega} + x_1^{i,\omega} + F_1^{i,\omega}(k_1^i) &= c_1^{i,\omega} + q^\omega(k_2^{i,\omega} - k_1^i) + m_2^\omega x_2^{i,\omega} \\ e_2^{i,\omega} + x_2^{i,\omega} + F_2^{i,\omega}(k_2^{i,\omega}) &= c_2^{i,\omega} \end{aligned} \tag{167}$$

Financial Constraint.

- date 0: borrower's security holdings x_1^b s.t. a convex set:

$$\Phi_1^b(x_1^b, k_1^b) \geq 0 \tag{168}$$

  interpretation:
  - $\Phi_1^b(x_1^b, k_1^b) := 0$: complete market
  - $\Phi_1^b(x_1^b, k_1^b) := (x_1^{b,\omega})_{\omega\in\Omega} = 0$: no financial trade
- date 1: borrower's security holdings $x_2^{b,\omega}$ s.t. a convex set [19]

$$\Phi_2^{b,\omega}(x_2^{b,\omega}, k_2^{b,\omega}; q^\omega) \geq 0 \tag{169}$$

  interpretation:
  - $\Phi_2^{b,\omega}() := x_2^{b,\omega} + \phi^\omega q^\omega k_2^{b,\omega} \geq 0$: partial collateralization of asset

Interpretations of financial constraint

- borrower = more productive entrepreneurs:
  financial constraint $\rightarrow$ inefficient sales of capital
- borrower = financial intermediary + firm:
  financial constraint $\rightarrow$ external finance $\downarrow$ $\rightarrow$ inefficient sales of capital
- borrower = homeowners holding mortgages:
  financial constraint $\rightarrow$ foreclosure $\rightarrow$ house depreciation $\uparrow$ $\rightarrow$ housing price $\downarrow$

[19] $\partial\Phi_2^{b,\omega}/\partial q^\omega \geq 0$: a higher price of the capital good *weakly* relaxes the financial constraint.

Decentralized Equilibrium: Date 1. Date 1 problem[20]:

$$V(n^{i,\omega}, k_1^i; N^\omega, K_1) = \max_{c_1\geq 0, c_2\geq 0, k_2, x_2} u(c_1^{i,\omega}) + \beta u(c_2^{i,\omega}) \tag{170}$$

s.t. two budget constraint (multiplier $\lambda_1^{i,\omega}$ and $\lambda_2^{i,\omega}$)

$$n^{i,\omega} \equiv e_1^{i,\omega} + x_1^{i,\omega} + F_1^{i,\omega}(k_1^i) = c_1^{i,\omega} + q^\omega(k_2^i - k_1^i) + m_2^\omega x_2^{i,\omega} \tag{171}$$

$$e_2^{i,\omega} + x_2^{i,\omega} + F_2^{i,\omega}(k_2^i) = c_2^{i,\omega} \tag{172}$$

a financial constraint (multiplier $\kappa_2^{b,\omega}$) [21]

$$\Phi_2^{b,\omega}(x_2^{b,\omega}, k_2^{b,\omega}; q^\omega) \geq 0 \tag{173}$$

Internalized Factors

- F.O.C. on security (debt)

$$m_2\lambda_1^i = \beta\lambda_2^i + \underbrace{\kappa_2^i(\partial\Phi_2^i/\partial x_2^i)}_{\text{shadow value of unit debt}} \tag{174}$$

  $\Rightarrow$ If financial constraint is slack, $\frac{\beta\lambda_2^l}{\lambda_1^l} = m_2$ (market discount).

  $\Rightarrow$ ... binding, $\frac{\beta\lambda_2^b}{\lambda_1^b} < m_2 \rightarrow$ capital value $\downarrow \rightarrow$ *fire sale discount*

- F.O.C. on capital

$$q\lambda_1^i = \beta\lambda_2^i F_2'(k_2^i) + \underbrace{\kappa_2^i(\partial\Phi_2^i/\partial k_2^i)}_{\text{benefit of relaxing constraint}} \tag{175}$$

  $\Rightarrow$ If ... binding, $\kappa_2^i\frac{\partial\Phi_2^i}{\partial k_2^i} > 0 \rightarrow$ capital value $\uparrow \rightarrow$ *collateral value*

- Equation (174) and (175) define price of bond (m_2) & capital (q)

Un-internalized Factors

- *un-internalized* welfare effects of sector-wide state N^ω [22]

$$\frac{dV^i}{dN^j} = \lambda_1^i \underbrace{[-\frac{\partial q^\omega}{\partial N^j}\Delta K_2^i - \frac{\partial m_2^\omega}{\partial N^j}X_2^i]}_{\equiv D^i_{N^j}(\textit{distributive effect})} + \kappa_2^i \underbrace{[\frac{\partial\Phi_2^i}{\partial q^\omega}\frac{\partial q^\omega}{\partial N^j}]}_{\equiv C^i_{N^j}(\textit{collateral effect})} \tag{176}$$

- *un-internalized* welfare effects of sector-wide state K_1

$$\begin{aligned}\frac{dV^i}{dK_1^j} =& \lambda_1^i \underbrace{[F'(K_1^i)D^i_{N^j} - \frac{\partial q^\omega}{\partial K_1^j}\Delta K_2^i - \frac{\partial m_2^\omega}{\partial K_1^j}X_2^i]}_{\equiv D^i_{K^j}(\textit{distributive effect})} \\ &+ \kappa_2^i \underbrace{[F_1'(K_1^i)C^i_{N^j} + \frac{\partial\Phi_2^i}{\partial q^\omega}\frac{\partial q^\omega}{\partial K_1^j}]}_{\equiv C^i_{K^j}(\textit{collateral effect})}\end{aligned} \tag{177}$$

Pecuniary Externalities

[20] Date 2 problem is trivial: agents consume and capital fully depreciates.

[21] multiplier of lender is $\kappa_2^{l,\omega} = 0$.

[22] In symmetric equilibrium, $N^i = n^i$, but individual agents take sector-wide state variable as given. Similarly, $K_1^i = k_1^i$.

- distributive effects
  - $(D^i_{N^j}, D^i_{K^j})$: j sector-wide state variables $N^j/K^j \to$ equilibrium price $\to$ marginal wealth redistribution towards sector i
  - zero-sum across all agents at *given* state

$$\sum_i D^i_{N^j} = 0 \quad \& \quad \sum_i D^i_{K^j} \tag{178}$$

- collateral effects
  - $(C^i_{N^j}, C^i_{K^j})$: j sector-wide state variables $N^j/K^j \to$ equilibrium price $\to$ tightness of borrowing constraint (faced by i)
  - generally not zero-sum across agents at *given* state
- source of pecuniary externalities
  - individual agents internalize $\partial V^i/\partial n^i \equiv \lambda^i_1$ and $\partial V^i/\partial k^i_1$
  - individual agents do not internalize $\partial V^i/\partial N^i$ and $\partial V^i/\partial K^i_1$

Decentralized Equilibrium: Date 0.

- optimization problem of agent

$$\max_{c_0, k_1, x_1} u(c_0) + \beta E_0[V^{i,\omega}(n^{i,\omega}, k^i_1; N^\omega, K_1)] \tag{179}$$

- s.t. budget constraint and financial constraint at date 0

$$e^i_0 = c^i_0 + h^i(k^i_1) + E_0[m^\omega_1 x^{i,\omega}_1] \tag{180}$$

$$\Phi^{b,\omega}_2(x^{b,\omega}_2, k^{b,\omega}_2; q^\omega) \geq 0 \tag{181}$$

- Euler equations (suppressing i, ω)[23]

$$m^\omega_1 \lambda_0 = \beta\lambda_1 + \kappa_1[\partial\Phi_1/\partial x_1] \tag{182}$$

$$h'(k_1)\lambda_0 = E_0[\beta\lambda_1(F^i_1)'(k_1) + q^\omega] + \kappa_1[\partial\Phi_1/\partial k_1] \tag{183}$$

Application 1

In this application, we try to answer the following question: Is the economy with fire sales always constrained inefficient ? As it turns out, an economy with fire sales can be constrained efficient, i.e., when risk markets are complete, or financial constraints do not depend on prices.

Assumptions.

- preference and endowment: no discount between date 1 and 2
- investment technology at date 0: $h(k_1) \to k_1$
  - borrowers: $h^b(k_1) = \alpha\frac{k^2}{2}$
  - lenders: $h^l(k_1) = +\infty \quad (\to k^l_1 = 0)$
- saving at date 0:
  - Arrow securities are available and no financial constraint $\Phi^b_1 \equiv 0$
  - risk market is complete
- production technology at date 1:
  - borrowers: $F^b_t(k) = A_t k$
  - lenders: $F^{'} l_t(0) = A_t$ and $F^{''} l_t(k) < 0$

[23]Similarly to date 1 problem, $\kappa_1 = 0$ implies $m^\omega_1 = \beta\lambda_1/\lambda_0$, i.e. intertemporal marginal rates of substitution of all agents are equalized absent of financial friction.

- financial constraint at date 1: (independent of q^ω)

$$\Phi_2^b(x_2^b, k_2^b) := x_2^b + \phi F_2^b(k_2^b), \quad \phi \in (0,1) \tag{184}$$

Problem at Date 1.

- "flow of resources": lenders → borrowers at date 1

$$z = m_2 x_2^l + q k_2^l \tag{185}$$

- "supply of fund": lenders gain at date 2

$$\rho(z) = x_2^l + F^l(k_2^l) \tag{186}$$

- resources given up by borrower:

$$\gamma(z) = x_2^l + A_2 k_2^l \tag{187}$$

- dead-weight loss of fire sales

$$\delta(z) = \gamma(z) - \rho(z) = A_2 k_2^l - F^l(k_2^l) \tag{188}$$

- market prices (pinned down by lender)

$$m_2 = \frac{\lambda_2^l}{\lambda_1^l} = \frac{u'(e_2^l + \rho(z))}{u'(n^l - z)} \tag{189}$$

$$q = m_2 F^{l'}(k_2^l) \tag{190}$$

- region 1: unconstrained equilibrium
  - slack financial constraint ⇒ no fire sales ⇒ $k_2^l = 0$
  - $z = m_2 x_2^l \Rightarrow \rho(z) = \gamma(z) = x_2^l \Rightarrow m_2 = \rho(z)/z \Rightarrow$

$$z u'(n^l - z) = \rho(z) u'(e_2^l + \rho(z))$$

    that defines a supply curve $\rho = \rho(z)$.
  - $\partial \rho / \partial z > 0$ under some conditions.
- region 2: constrained equilibrium (w. fire sales)
  - binding financial constraint ⇒ $x_2^l = m_2 \phi A_2 (k_1^b - k_2^l) \Rightarrow$

$$z u'(n^l - z) = u'(e_2^l + \phi A_2(k_1^b - k_2^l) + F^l(k_2^l))[\phi A_2(k_1^b - k_2^l) + k_2^l F^{l'}(k_2^l)]$$

    that defines a "demand" curve for fire sales $k_2^l = k(z)$
  - $\partial k / \partial z > 0$ under some conditions.
  - $\rho(z)$ is strictly increasing with z.

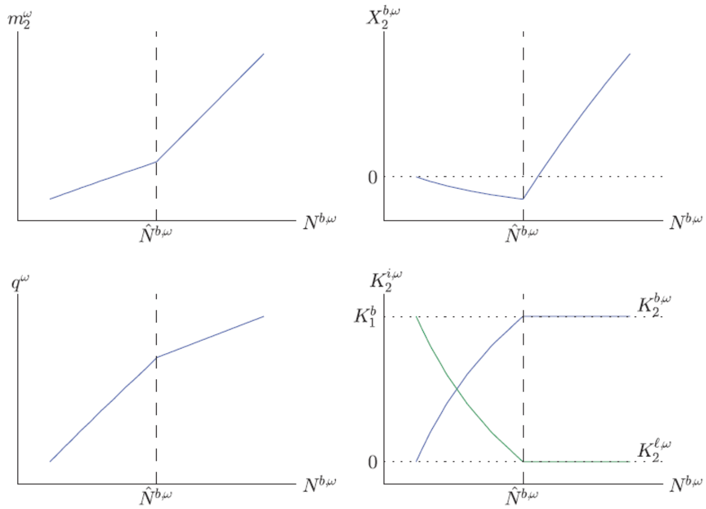


FIGURE 1
Date 1 equilibrium

FIGURE 26. Date 1 Equilibrium

Constrained Efficiency.

- problem at date 0: ($MRS^{i,\omega} = \beta\lambda_1^{i,\omega}/\lambda_0^i$)

$$\Delta MRS^{ij,\omega} = MRS^{i,\omega} - MRS^{j,\omega} = 0 \tag{191}$$

  $\rightarrow N^b > \hat{N}^b$: slack financial constraint $\rightarrow$ first-best allocation
  $\rightarrow N^b < \hat{N}^b$: binding financial constraint $\rightarrow$ constrained efficient
- collateral externality: absent from financial constraint (184)
- distributive externality: absent due to complete risk market
  - (b/w date 0 and 1) agents optimally share risks before fire sales $\rightarrow \Delta MRS^{ij,\omega} = 0$
  - (b/w date 1 and 2) given ω agent's welfare is monotonic with z $\rightarrow$ any change in flow of resources z hurts one party.
  - no scope for welfare improvement using distributive measures.
- this decentralized equilibrium with fire sales: constrained efficient

Lesson from Application 1

- sign (magnitude) of distributive externalities
  $\Leftarrow$ product of 3 (sufficient) variables
  - D1: difference in MRS of agents ($\Delta MRS^{ij,\omega}$)
  - D2: trading position of capital and financial assets ($\Delta k^{i,\omega}$)
  - D3: sensitivity of eqm price to sectoral state variables ($\frac{\partial q^\omega}{\partial N^{b,\omega}}, \frac{\partial q^\omega}{\partial K^b}$)
- In Application 1: signs of D2 and D3 become irrelevant as D1 = 0.
- In Application 2: sign of D2 can be either + or -.
- In Application 3: sign of D1 can be either + or -.

Application 4

In this application, we try to answer the following question: Is collateral externality always consistent with over-borrowing? Is collateral externality always consistent with over-investment? As a preview, collateral externality is always consistent with over-borrowing, but it can be consistent with both over-investment and under-investment.
Assumptions.

- preference and endowment: no time discount
  - lender's preference: $U^l = c_0^l + c_1^l + c_2^l$
  - lender's endowment: $e_t^l = +\infty \quad (\to m_2 = 1)$
  - borrower's preference: $U^b = \log c_0^b + \log c_1^b + c_2^b$
  - borrower's endowment: $1 \geq e_0^b > e_1^b = e_2^b = 0$
- investment technology at date 0: $h(k_1) \to k_1$
  - borrowers: $h^b(k_1) = \alpha \frac{k^2}{2}$
  - lenders: $h^l(k_1) = +\infty \quad (\to k_1^l = 0)$
- perfect foresight economy with no uncertainty:
  - $\equiv$ complete risk market
- production technology at date 1:
  - borrowers: $F_t^b(k) = A_t k$, with $\alpha > A_1 + A_2 > 0$ and $A_2 > 0$ [24]
  - lenders: $F_t^l(k) = 0 \quad (\to k_2^l = 0)$
- distributive externalities ($D^i_{N^j}$ and $D^i_{K^b}$) are zero by assumption
  - constant bond price: $m_2 = 1$
  - no capital trade *between sectors*: $k_t^l = 0$
- we focus on collateral externalities
  - at date 0 no financial friction
  - at date 1 borrowers can borrow up to ϕ fraction of asset value [25]

$$\Phi_2^b(x_2^b, k_2^b; q) : x_2^b + \phi q k_2^b \geq 0 \tag{192}$$

Problem at Date 1.

- lenders: $m_2 = 1$
- borrowers:

$$V^b(n^b, k_1^b; N, K_1) = \max_{x_2^b, k_2^b} u(n^b - q\Delta k_2^b - x_2^b) + x_2^b + A_2 k_2^b + \kappa_2^b(x_2^b + \phi q k_2^b) \tag{193}$$

  f.o.c. w.r.t k_2^b and x_2^b

$$q[u'(c_1^b - \phi\kappa_2^b)] = A_2 \tag{194}$$

$$u'(c_1^b) = 1 + \kappa_2^b \tag{195}$$

- capital price ($q(C_1^b)$) in equilibrium:

$$q = \frac{A_2 C_1^b}{1 - \phi + \phi C_1^b} \tag{196}$$

- first-best allocation:
  - $C_0^b = C_1^b = 1$ and $K_t^{b*} = \frac{A_1 + A_2}{\alpha}$
  - feasible if $X_2^b \geq -\phi q K_2^b$, or $N^b \in [1 - \phi A_2 K_1^b, +\infty)$
- we focus on constrained equilibrium:
  - $N^b \in (0, 1 - \phi A_2 K_1^b)$

[24] We allow A_1 to be negative, capturing maintenance cost of capital.
[25] We assume ϕ is less than $\frac{1}{1+A_2}$, a property we use later.

- binding financial constraint: $X_2^b = -\phi q K_2^b$
- budget constraint implies a unique $C_1^b = C_1^b(N^b, K_1^b)$ from

$$C_1^b = N^b + \phi q K_1^b = N^b + \phi K_1^b \frac{A_2 C_1^b}{1 - \phi + \phi C_1^b} \tag{197}$$

- consumption $C_1^b(N^b, K_1^b)$ increases in both N^b and K_1^b
- price of capital, $q(C_1^b(N^b, K_1^b))$, increases in both N^b and K_1^b.

Collateral Externality.

- collateral externality:

$$C_{N^b}^b = \phi K_1^b \frac{\partial q}{\partial N^b} > 0 \tag{198}$$

$$C_{K_1^b}^b = \phi K_1^b (A_1 \frac{\partial q}{\partial N^b} + \frac{\partial q}{\partial K_1^b}) = \frac{\phi K_1^b q'(C_1^b)}{1 - \phi K_1^b q'(C_1^b)} (A_1 + \phi q) \tag{199}$$

- sign of collateral externality $C_{N^b}^b$: positive
  - $\Rightarrow$ borrowers engage in over-borrowing
  - planner: saving $\uparrow \Rightarrow$ net worth $\uparrow \Rightarrow$ q $\uparrow \Rightarrow$ financial constraint $\downarrow$
- sign of collateral externality $C_{K_1^b}^b$ = sign of $(A_1 + \phi q)$
  - $A_1 < -\phi q$: capital $\uparrow \rightarrow$ liquid net worth of borrower *sector* $\downarrow \rightarrow$ q $\downarrow \rightarrow$ negative collateral effect
  - cut-off $\hat{A}_1 : \hat{A}_1 + \phi q(\hat{A}_1) = 0$

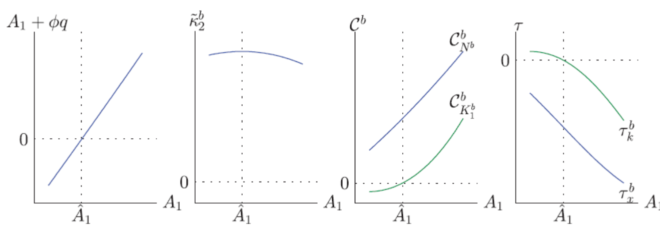


FIGURE 4
Components of optimal taxes τ_x^b, τ_k^b in Application 4

FIGURE 27. Comparative Statics

Comparative Statics.

- borrowers over-invest if $A_1 < \hat{A}_1$
- borrowers invest efficiently if $A_1 = \hat{A}_1$
- borrowers under-invest if $A_1 > \hat{A}_1$

Lesson from Application 4

- sign (magnitude) of collateral externalities
  $\Leftarrow$ product of 3 (sufficient) statistics
  - C1: shadow value of binding financial constraint $(\kappa > 0)$
  - C2: sensitivity of financial constraint to asset price $(\frac{\partial \Phi_2^\omega}{\partial q^\omega} > 0)$
  - C3: sensitivity of eqm price to sectoral state variables $(\frac{\partial q^\omega}{\partial N^{b,\omega}}, \frac{\partial q^\omega}{\partial K^b})$
- In Application 4: signs of $C_{K^b}^b$ vary with A_1 while $C_{N^b}^b > 0$

$$\tau_x^{i,\omega} = -\Delta MRS^{ij,\omega}\mathcal{D}_{N^i}^{i,\omega} - \tilde{\kappa}_2^{b,\omega}\mathcal{C}_{N^i}^{b,\omega}, \forall i,\omega$$
$$\tau_k^{i} = -\mathbb{E}_0\left[\Delta MRS^{ij,\omega}\mathcal{D}_{K^i}^{i,\omega}\right] - \mathbb{E}_0\left[\tilde{\kappa}_2^{b,\omega}\mathcal{C}_{K^i}^{b,\omega}\right], \forall i$$

7.2.1. *Policy: Corrective Tax.*

- positive $\tau_x^{i,\omega}$ tax: agent i should carry less wealth toward ω
- negative τ_k^i tax: agent i should invest less in capital
- examples:
  - distributive externality: $\Delta k^{b,\omega} < 0$, $\Delta MRS^{bl,\omega} > 0$, $\frac{\partial q^\omega}{\partial N^{b,\omega}} > 0$
    $\Rightarrow \tau_x^{b,\omega} < 0$: borrowers under-save
  - collateral externality: $\kappa^{b,\omega} > 0$, $\frac{\partial \Phi_2^\omega}{\partial q^\omega} > 0$, $\frac{\partial q^\omega}{\partial N^{b,\omega}} > 0$
    $\Rightarrow \tau_x^{b,\omega} < 0$: borrowers under-save

8. Incomplete Market and Household Heterogeneity

8.1. *** Krusell and Smith (1988, JPE).** Krusell and Smith (1988) is the workhorse model of household heterogeneity. It is a popular model with a continuum of agents, idiosyncratic income shocks, incomplete financial markets, and aggregate uncertainty.

8.1.1. *Model.* The economy is a production economy with aggregate shocks in which agents face different employment histories and partially insure themselves through (dis)saving in capital. An inequality constraint prevents agents from borrowing, i.e., taking short positions in capital. The model is identical to the benchmark model in Krusell and Smith (1998), except that we introduce unemployment benefits; without unemployment benefits the borrowing constraint would never be binding.
Problem for the individual agent. The economy consists of a unit mass of ex ante identical households. Each period, agents face an idiosyncratic shock ε that determines whether they are employed, $\varepsilon = 1$, or unemployed, $\varepsilon = 0$. An employed agent earns a wage rate of w_t and an after-tax wage rate of $(1-\tau_t)\,w_t$. An unemployed agent receives unemployment benefits μw_t. Note that Krusell and Smith (1998) set μ equal to zero. This is the only difference with their model. Markets are incomplete and the only investment available is capital accumulation. The net rate of return on this investment is equal to $r_t - \delta$, where r_t is the rental rate and δ is the depreciation rate. Agent's i maximization problem is as follows:

$$\begin{aligned}
&\max_{\left\{c_t^i, k_{t+1}^i\right]_{z=0}^{\infty}} E\sum_{t=0}^{\infty}\beta^t \frac{\left(c_t^i\right)^{1-\gamma}-1}{1-\gamma}\\
&\text{s.t.} \quad c_t^i + k_{t+1}^i = r_t k_t^i + \left[(1-\tau_t)\,\bar{l}\varepsilon_t^i + \mu\left(1-\varepsilon_t^i\right)\right] w_t + (1-\delta)k_t^i\\
&k_{t+1}^i \geq 0
\end{aligned}$$

Here c_t^i is the individual level of consumption, k_t^i is the agent's beginning-of-period capital, and $\bar{l}$ is the time endowment.
Firm problem. Markets are competitive and the production technology of the firm is characterized by a Cobb-Douglas production function. Consequently, firm heterogeneity is not an issue. Let K_t and L_t stand for per capita capital and the employment rate, respectively. Per capita output is given by

$$Y_t = a_t K_t^{\alpha}\left(\bar{L}_t\right)^{1-\alpha}$$

and prices by

$$\begin{aligned}
w_t &= (1-\alpha)a_t\left(\frac{K_t}{\bar{l}L_t}\right)^{\alpha} \text{ and}\\
r_t &= \alpha a_t\left(\frac{K_t}{\bar{l}L_t}\right)^{\alpha-1}
\end{aligned}$$

Aggregate productivity, a_t, is an exogenous stochastic process that can take on two values, $1-\Delta_a$ and $1+\Delta_a$.
Government. The only role of the government is to tax employed agents and to redistribute funds to the unemployed. We assume that the government's budget is balanced each period. This implies that the tax rate is equal to

$$\tau_t = \frac{\mu u_t}{\bar{L}_t}$$

where $u_t = 1 - L_t$ denotes the unemployment rate in period t

Exogenous driving processes. There are two stochastic driving processes. The first is aggregate productivity and the second is the employment status. Both are assumed to be first-order Markov processes. We let $\pi_{a\alpha' cc'}$ stand for the probability that $a_{t+1} = a'$ and $\varepsilon^i_{t+1} = \varepsilon'$ when $a_t = a$ and $\varepsilon^i_t = \varepsilon$. These transition probabilities are chosen such that the unemployment rate is a function of a only and can, thus, take on only two values. That is, $u_t = u\,(a_t)$ with $u_b = u\,(1 - \Delta_a) > u_g = u\,(1 + \Delta_a)$

8.2. *** Auclert (2019, AER).** This paper studies the role of redistribution in the transmission mechanism of monetary policy to consumption. Three key channels highlighted in the paper are earnings heterogeneity channel (income Y), Fisher channel (price level P), and interest rate exposure channel (real interest rate R).

8.2.1. *Introduction.* Classics channels of MP:

- Interest Rate Channel
  i $\downarrow \Rightarrow$ r $\downarrow$ (with sticky price) $\Rightarrow$ c $\uparrow$ (*Euler Equation*)
- Exchange Rate Channel
  i $\downarrow \Rightarrow \varepsilon \downarrow \Rightarrow$ NX $\uparrow \Rightarrow$ Y $\uparrow \Rightarrow$ C $\uparrow$
- Asset Price Channel[26] (*q-theory & life-cycle theory*)
  i $\downarrow \Rightarrow$ equity > debt $\Rightarrow$ equity price $\uparrow \Rightarrow$ wealth $\uparrow \Rightarrow$ c $\uparrow$
- Credit Channel
  - Bank Lending Channels
    i $\downarrow \Rightarrow$ bank's lending$\Rightarrow$ firm's borrowing
  - Balance Sheet Channels (BG, 1995)
    i $\downarrow \Rightarrow$ balance sheet improves $\Rightarrow$ cost of borrowing $\downarrow$

Effects of monetary expansions:

- increase real income (from labor/capital)
- raise inflation
- lower real interest rates

not everyone is equally affected by these changes:

- working hours and capital ownership is unlikely to be equal
- unexpected inflation revalues nominal balance sheets.
  $\rightarrow$ *nominal* creditors lose and *nominal* debtors gain.
- lower R doesn't necessarily benefit asset holders
  $\rightarrow$ *duration* and *measurement* of assets and liabilities matter

Channels of MP on consumption revisited (* denotes redistribution channels):

- Interest Rate Channel*
  un-hedged interest rate exposure, URE
- Earning Heterogeneity Channel*
- Fisher Channel*
  net nominal position, NNP
- Income Channel
- Substitution Channel
- Exchange Rate Channel
  closed economy
- Asset Price Channel
  secondary effect through dY, dP and dR

8.2.2. *PE Model 1.* In this section we introduce a partial equilibrium model with the following assumptions:

- complete market
- no uncertainty
- separable preference over c and n

[26] Iacaviello (2005) etc. also explore the real estate price channel.

- perfect foresight over P and W

We consider three types of agent:

- type (1): with no financial asset
- type (2): with real bond
- type (3): with nominal and real bond

UMP of agent (1) with no financial asset. In each period t, the agent solves

$$\begin{aligned} max \quad & \sum_t \beta^t \{u(c_t) - v(n_t)\} \\ s.t. \quad & P_t c_t = P_t y_t + W_t n_t \end{aligned}$$

where $P_t y_t$ is endowed income, aka claimed profit, and $W_t n_t$ is wage income.
At period 0, the life-time utility maximization problem is

$$\begin{aligned} max \quad & \sum_t \beta^t \{u(c_t) - v(n_t)\} \\ s.t. \quad & \sum_{t\geq 0} c_t = \sum_{t\geq 0} (y_t + w_t n_t) \end{aligned}$$

where w=W/P is real wage rate.
UMP of agent (2) with real bond. In each period t, the agent solves

$$max \quad \sum_t \beta^t \{u(c_t) - v(n_t)\}$$

s.t.

$$\begin{aligned} P_t c_t + \sum_{s\geqslant 1} {}_t q_{t+s} ({}_t b_{t+s}) P_{t+s} = P_t y_t + W_t n_t + \\ ({}_{t-1} b_t) P_t + \sum_{s\geqslant 1} {}_t q_{t+s} ({}_{t-1} b_{t+s}) P_{t+s} \end{aligned}$$

where ${}_t q_{t+s}$ is time-t (real) price of real zero coupon bonds that mature at time t+s, and ${}_t b_{t+s}$ is the quantity purchased. For exposition we denote ${}_0 q_t \equiv q_t$.
At period 0, the life-time utility maximization problem is

$$\begin{aligned} max \quad & \sum_t \beta^t \{u(c_t) - v(n_t)\} \\ s.t. \quad & \sum_{t\geq 0} q_t c_t = \sum_{t\geq 0} q_t (y_t + w_t n_t) + \sum_{t\geq 0} q_t ({}_{-1} b_t) \end{aligned}$$

UMP of Agent (3) with with real and nominal bond. In each period t, the agent solves

$$max \quad \sum_t \beta^t \{u(c_t) - v(n_t)\}$$

$$\begin{aligned} s.t. \quad P_t c_t + \sum_{s\geq 1} {}_t Q_{t+s} \; {}_t B_{t+s} + \sum_{s\geq 1} {}_t q_{t+s} \; {}_t b_{t+s} \, P_{t+s} = P_t y_t + W_t n_t + \\ ({}_{t-1} B_t) + \sum_{s\geq 1} {}_t Q_{t+s} \; {}_{t-1} B_{t+s} + ({}_{t-1} b_t) P_t + \sum_{s\geq 1} {}_t q_{t+s} \; {}_{t-1} b_{t+s} \, P_{t+s} \end{aligned} \tag{200}$$

where ${}_t Q_{t+s}$ is time-t price of nominal zero coupon bonds that mature at time t+s, and ${}_t B_{t+s}$ is the quantity purchased.

No Arbitrage Condition. At period t, with 1 dollar:

- The nominal return to nominal bonds that mature at period t+s:

$$\frac{1}{{}_tQ_{t+s}}$$

- The nominal return to real bonds that mature at period t+s:

$$\frac{1}{{}_tq_{t+s}\,P_t}P_{t+s}$$

- $\Rightarrow$ No arbitrage condition (i.e. fisher equation):

$$ {}_tQ_{t+s} = ({}_tq_{t+s})\frac{P_t}{P_{t+s}} \tag{201}$$

Real Flow Budget Constraint.

- Nominal (Q replaced by q):

$$P_tc_t + \sum_{s\geqslant 1} ({}_tq_{t+s})\frac{P_t}{P_{t+s}}\,{}_tB_{t+s} + \sum_{s\geqslant 1}\,{}_tq_{t+s}\ {}_tb_{t+s}\,P_{t+s} = P_ty_t + W_tn_t +$$

$$({}_{t-1}B_t) + \sum_{s\geqslant 1} ({}_tq_{t+s})\frac{P_t}{P_{t+s}}\,{}_{t-1}B_{t+s} + ({}_{t-1}b_t)P_t + \sum_{s\geqslant 1}\,{}_tq_{t+s}\ {}_{t-1}b_{t+s}\,P_{t+s}$$

- Real:

$$c_t + \sum_{s\geqslant 1}({}_tq_{t+s})\frac{1}{P_{t+s}}\,{}_tB_{t+s} + \sum_{s\geqslant 1}\,{}_tq_{t+s}\ {}_tb_{t+s}\,\frac{P_{t+s}}{P_t} = y_t + w_tn_t +$$

$$\frac{{}_{t-1}B_t}{P_t} + \sum_{s\geqslant 1}({}_tq_{t+s})\frac{1}{P_{t+s}}\,{}_{t-1}B_{t+s} + ({}_{t-1}b_t) + \sum_{s\geqslant 1}\,{}_tq_{t+s}\ {}_{t-1}b_{t+s}\,\frac{P_{t+s}}{P_t}$$

At period 0, the life-time utility maximization problem is

$$max \quad \sum_t \beta^t\{u(c_t) - v(n_t)\}$$

$$s.t. \quad \sum_{t\geq 0} q_tc_t = \underbrace{\sum_{t\geq 0} q_t[y_t + w_tn_t]}_{\omega^H:\ \text{human wealth}} + \underbrace{\sum_{t\geq 0} q_t[({}_{-1}b_t) + (\frac{{}_{-1}B_t}{P_t})]}_{\omega^F:\ \text{financial wealth}} \equiv \omega \tag{202}$$

- Message: financial assets with same financial wealth deliver same solution to UMP. $\Rightarrow$ the composition of balance sheet is irrelevant.
- Question: Is the composition relevant after a shock?

Benchmark: MP Shock in NK Models. A stylized NK model with no uncertainty and investment features:

$$log(\frac{c_t}{\bar{c}}) = log(\frac{c_{t+1}}{\bar{c}}) - \sigma(i_t - log(\frac{P_{t+1}}{P_t}) - \varrho) \tag{203}$$

$$log(\frac{P_t}{P_{t-1}}) = \beta log(\frac{P_{t+1}}{P_t)} + \kappa log(\frac{c_t}{\bar{c}}) \tag{204}$$

$$i_t = \varrho + \phi_\pi log(\frac{c_t}{\bar{c}}) + \varepsilon_t \tag{205}$$

Now consider a one-time monetary shock:

$$\varepsilon_0 < 0; \quad and \quad \varepsilon_t = 0 \quad \forall t \neq 0 \tag{206}$$

where $\bar{x}$ is steady state value of x; $\varrho = 1/\beta - 1$ is steady state real interest rate; σ is the elasticity of substitution; κ is f(parameter).
The solution features:

$$i_t = \varrho; \quad P_t = P_{t-1} \quad c_t = \bar{t} \quad \forall t \geqslant 1$$

solving it backward, impact on i and c is one-shot ($i_0 \downarrow$, $c_0 \uparrow$);

$$i_0 = \varrho + \frac{1}{1 + \kappa\sigma\phi_\pi}\varepsilon_0$$

$$log(\frac{c_0}{\bar{c}}) = -\frac{\sigma}{1 + \kappa\sigma\phi_\pi}\varepsilon_0$$

Impact on P is immediate and permanent ($P_t \uparrow$):

$$log(\frac{P_0}{\bar{P}}) = -\frac{\kappa\sigma}{1 + \kappa\sigma\phi_\pi}\varepsilon_0$$

Given that wage

$$w_t = \frac{v'(c_t^{1/1-\alpha})}{u'(c_t)}$$

$\rightarrow$ The impact on *wage* is one-shot ($w_0 \uparrow$).
Given that capital rent

$$\rho_t = \frac{\alpha}{1-\alpha} w_t c_t^{1/1-\alpha} = \frac{\alpha}{1-\alpha}\frac{v'(c_t^{1/1-\alpha})}{u'(c_t)} c_t^{1/1-\alpha}$$

$\rightarrow$ The impact on *capital return* is one-shot ($\rho_0 \uparrow$).
Given that claimed profit

$$\pi_t = c_t - w_t n_t - \rho_t k = c_t(1 - \frac{\alpha}{1-\alpha}\frac{v'(c_t^{1/1-\alpha})}{u'(c_t)} c_t^{1/1-\alpha})$$

$\rightarrow$ The impact on *claimed profit* is one-shot ($\pi_0 \uparrow$).
Given that $q_0 = Q_0 = 1$, and $P_t = P_0$ for $t \geqslant 1$,

$$q_t = Q_t = \Pi_{s=0}^{t-1}\,{}_sQ_t = \frac{1}{1+i_0}\beta^{t-1}$$

where the first equation utilizes no arbitrage condition that

$$Q_t = q_t \frac{P_0}{P_t}$$

Define R=1+i, we have that for $t \geqslant 1$.

$$\frac{dq_t}{q_t} = \frac{dQ_t}{Q_t} = -\frac{dR_0}{R_0},$$

$\rightarrow$ The impact on *nominal and real state prices* is permanent, starting from t=1.

Transitory Monetary Shock. Keep balance sheet fixed at $\{{}_{-1}B_t\}_{t\geqslant 0}$, $\{{}_{-1}b_t\}_{t\geqslant 0}$, a stylized transitory monetary policy shock at period 0 in New Keynesian models features:

- Nominal price rises in proportion after period 0; $\frac{dP_t}{P_t} = \frac{dP}{P}$, for $t \geqslant 0$.
- Present-value discount rate rises in proportion after period 1; $\frac{dq_t}{q_t} = -\frac{dR}{R}$, for $t \geqslant 1$.
- Fisher equation holds again after period 1; $\frac{dQ_t}{Q_t} = -\frac{dR}{R}$, for $t \geqslant 1$.
- Endowed income and real wage rise at period 0 only.
- Impact on consumption and interest rate at period 0 only.

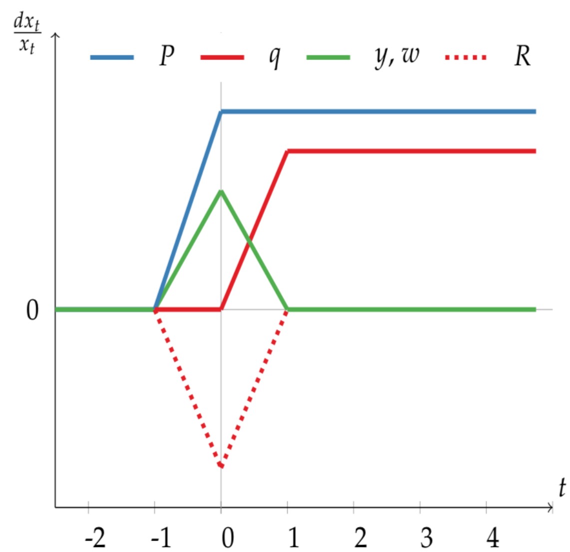


FIGURE 28. Impulse response to a transitory monetary policy shock.

This paper is interested in the first-order change in initial consumption ($dc = dc_0$), labor supply ($dn = dn_0$) and welfare (dU) after the monetary policy shock.

Theorem 1.

$$dc = MPC(d\Omega + \psi n dw) - \sigma c MPS\frac{dR}{R} \tag{207}$$

$$dn = MPN(d\Omega + \psi n dw) + \psi n MPS\frac{dR}{R} + \psi n\frac{dw}{w} \tag{208}$$

$$dU = u'(c)d\Omega \tag{209}$$

where $d\Omega$ is *net-of-consumption wealth change*, $MPC = \partial c_0/\partial y_0$; $MPN = \partial n_0/\partial y_0$; $MPS = 1 - MPC + w_0 MPN$.

Interpretation of Theorem 1.

We start by unpacking wealth effect ($d\Omega$), aggregates net-of-consumption wealth change, which shows up in

$$dc = MPC(d\Omega + \psi n dw) - \sigma c MPS\frac{dR}{R}$$

$$dn = MPN(d\Omega + \psi n dw) + \psi n MPS\frac{dR}{R} + \psi n\frac{dw}{w}$$

$$dU = u'(c)d\Omega$$

We first decompose $d\Omega =$

$$\underbrace{dy + ndw}_{Earning} - \underbrace{\sum_{t\geq 0} Q_t(\frac{{}_{-1}B_t}{P_0})\frac{dP}{P}}_{Fisher} + \underbrace{(y + wn + (\frac{{}_{-1}B_t}{P_0}) + ({}_{-1}b_t) - c)\frac{dR}{R}}_{URE}$$

$$\underbrace{dy + ndw}_{Earning} - \underbrace{\sum_{t\geq 0} Q_t(\frac{{}_{-1}B_t}{P_0})\frac{dP}{P}}_{Fisher} + \underbrace{(y + wn + (\frac{{}_{-1}B_t}{P_0}) + ({}_{-1}b_t) - c)\frac{dR}{R}}_{URE}$$

which features three major channels:

- **Earning Channel**: Monetary policy affects present value of income, a sum of endowed income and wage income.
  Working hours, n, measures exposure of workers to wage change, i.e., the more he works, the more he benefits.

$$\underbrace{dy + ndw}_{Earning} - \underbrace{\sum_{t\geqslant 0} Q_t(\frac{{}_{-1}B_t}{P_0})\frac{dP}{P}}_{Fisher} + \underbrace{(y + wn + (\frac{{}_{-1}B_t}{P_0}) + ({}_{-1}b_t) - c)\frac{dR}{R}}_{URE}$$

- **Fisher Channel**: Monetary policy affects nominal price level (immediately and permanently), generating nominal denomination of assets and liabilities.
  Net Nominal Position (**NNP**): Present value of *nominal* assets.

$$dy + ndw - \underbrace{\sum_{t\geqslant 0} Q_t(\frac{{}_{-1}B_t}{P_0})}_{NNP}\frac{dP}{P} + (y + wn + (\frac{{}_{-1}B_t}{P_0}) + ({}_{-1}b_t) - c)\frac{dR}{R}$$

$$\underbrace{dy + ndw}_{Earning} - \underbrace{\sum_{t\geq 0} Q_t(\frac{{}_{-1}B_t}{P_0})\frac{dP}{P}}_{Fisher} + \underbrace{(y + wn + (\frac{{}_{-1}B_t}{P_0}) + ({}_{-1}b_t) - c)\frac{dR}{R}}_{URE\quad Channel}$$

- **Interest Rate Exposure Channel**: Monetary policy affects real interest rate.
  - Unhedged Interest Rate Exposure (**URE**): The difference between all maturing assets(including income) and liabilities (including planned consumption) at time 0.

$$dy + ndw - \sum_{t\geq 0} Q_t(\frac{{}_{-1}B_t}{P_0})\frac{dP}{P} + \underbrace{(y + wn + (\frac{{}_{-1}B_t}{P_0}) + ({}_{-1}b_t) - c)}_{URE}\frac{dR}{R}$$

  - Suppose that $dR < 0 \Leftrightarrow$ a *decline* in the discount rate:
    $\Rightarrow$ Present value of *future* assets $\uparrow$ (*traditional capital gain view*)
    $\Rightarrow$ Present value of *future* liabilities $\uparrow$
    $\Leftrightarrow$ net wealth gain iff *future* assets $>$ *future* liabilities

$$\sum_{t\geq 1} q_t[y_t + w_t n_t] + \sum_{t\geq 1} q_t[({}_{-1}b_t) + (\frac{{}_{-1}B_t}{P_t})] > \sum_{t\geq 1} q_t c_t$$

given that *lifetime* assets = *lifetime* liability

$$\sum_{t\geq 0} q_t[y_t + w_t n_t] + \sum_{t\geq 0} q_t[({}_{-1}b_t) + (\frac{{}_{-1}B_t}{P_t})] = \sum_{t\geq 0} q_t c_t$$

$\Leftrightarrow$ net wealth gain iff *current* assets < *current* liability, aka:

$$URE = y + wn + (\frac{{}_{-1}B_t}{P_0}) + ({}_{-1}b_t) - c < 0$$

- Implication: duration of asset plan matters after interest rate shock
  - Fixed-Rate Mortgage holders/ Annuitized Retirees:
    $URE = 0 \Rightarrow$ income and outlays roughly balanced
  - Adjustable-Rate Mortgage Holders:
    $URE < 0 \Rightarrow$ *gain* from temporary interest rate decline
  - Savers with large amount of short-duration wealth:
    $URE > 0 \Rightarrow$ *lose* from temporary interest rate decline

- **Decomposition**: net of consumption wealth change $d\Omega$ as:

$$d\Omega = dy + ndw - NNP\frac{dP}{P} + URE\frac{dR}{R} \tag{210}$$

Discussion: monetary policy and household welfare.

- Popular discussion: Asset value affects welfare of holders.
  $\Rightarrow$ mp $\rightarrow$ i $\downarrow$ $\rightarrow$ bond price $\uparrow$ $\rightarrow$ bond holders benefit
- *Our model:* mp does *not* affect asset values directly
  Monetary policy influence asset values through three channels: a risk-free real discount rate effect (dR), an inflation effect (dP), and an effect on dividends (dy).

$$dU = u'(c)d\Omega = u'(c)(dy + ndw - NNP\frac{dP}{P} + URE\frac{dR}{R}) \tag{211}$$

- Benefit long-term bond holders with short-term consumption
- Hurt short-term bond holders with long-term consumption
  i.e., by lowering return to re-investment of wealth.

Theorem 1 revisited: interest rate.

$$dc = MPC(d\Omega + \psi ndw) - \sigma cMPS\frac{dR}{R}$$

$$dc = MPC(dy + ndw - NNP\frac{dP}{P} + URE\frac{dR}{R} + \psi ndw) - \sigma cMPS\frac{dR}{R}$$

$$dc = MPC(dy + ndw - NNP\frac{dP}{P} + \psi ndw) + (MPC * URE - \sigma cMPS)\frac{dR}{R}$$

$\Rightarrow$ A decline in interest rate increases consumption iff

$$\sigma cMPS > MPC * URE$$

$\Leftrightarrow$ substitution effect > income effect
now define dY as *overall* change in income:

$$dY = dy + ndw + wdn$$

Corollary 1 *(overall response of consumption)*

$$dc = M\hat{P}C(dY - NNP\frac{dP}{P} + URE\frac{dR}{R}) - \sigma c(1 - M\hat{P}C)\frac{dR}{R} \tag{212}$$

where $\hat{MPC} = \frac{MPC}{MPC+MPS} = \frac{MPC}{1+wMPN} \geqslant MPC$.

$$dc = \underbrace{\hat{MPC} * dY}_{\text{aggregate income}} - \underbrace{\hat{MPC} * NNP\frac{dP}{P}}_{Fisher} + \underbrace{\hat{MPC} * URE\frac{dR}{R}}_{URE} - \underbrace{\sigma c\hat{MPS}\frac{dR}{R}}_{Substitution}$$

where $\hat{MPS} = 1 - \hat{MPC} = \frac{MPS}{MPC+MPS}$.
Extensions. There are a few extensions from PE model 1.

- utility function: separable ⇒ general
- consumption goods: non-durable ⇒ non-durable and durable
- complete market ⇒ incomplete market with uninsured risk (PE Model 2)

PE Model 2

In this section we extend PE model 1 by relaxing the assumption on complete market. In specific, we assume

- incomplete market
  - limited set of assets can be traded
  - borrowing constraint
- agent can trade in N stocks:
  - real price $S_t = (S_{1t}, S_{2t}, ...S_{Nt})$
  - pay real dividends $d_t = (d_{1t}, d_{2t}, ...d_{Nt})$
  - portfolio of share: $\theta_t = (\theta_{1t}, \theta_{2t}, ...\theta_{Nt})$
- agent can trade in long-term bond:
  - nominal price Q_t at time t
  - pay declining nominal coupon $(1, \delta, \delta^2...)$ from period t+1
  - current bond portfolio Λ_t at time t
- idiosyncratic income uncertainty
- separable preference over c and n

UMP of Agent:

$$max \quad E[\sum_t \beta^t\{u(c_t) - v(n_t)\}]$$

s.t. budget constraint:

$$P_t c_t + Q_t(\Lambda_{t+1} - \delta\Lambda_t) + \theta_{t+1}P_t S_t =$$

$$P_t y_t + P_t w_t n_t + \Lambda_t + \theta_t(P_t S_t + P_t d_t) \tag{213}$$

borrowing constraint *(end of period wealth cannot be too negative)*

$$\frac{Q_t\Lambda_t + \theta_{t+1}P_t S_t}{P_t} \geq -\frac{\bar{D}}{R_t} \tag{214}$$

Channels of monetary policy:

- NNP (Net Nominal Position)

$$NNP_t \equiv \underbrace{\frac{\Lambda_t}{P_t}}_{\text{current}} + \underbrace{Q_t\delta\frac{\Lambda_t}{P_t}}_{\text{PV of future}}$$

- URE (Un-hedged Interest Rate Exposure)

$$URE_t \equiv \underbrace{y_y + w_t n_t + \frac{\Lambda_t}{P_t} + \theta_t d_t}_{\text{maturing assets}} - \underbrace{c_t}_{liabilities}$$

Theorem 2. Assume that the consumers is

- *at interior optimum*, or
- *at a binding borrowing constraint*, or
- *unable to access financial market*

then

$$dc = MPC(d\Omega + \psi n dw) - \sigma c MPS \frac{dR}{R} \tag{215}$$

$$dc = M\hat{P}C(dY - NNP\frac{dP}{P} + URE\frac{dR}{R}) - \sigma c(1 - M\hat{P}C)\frac{dR}{R} \tag{216}$$

$$dn = MPN(d\Omega + \psi n dw) + \psi n MPS\frac{dR}{R} + \psi n\frac{dw}{w} \tag{217}$$

- When *at interior optimum*
  ⇒ ∼ Theorem 1
- When *at a binding borrowing constraint*
  ⇒ change in borrowing capacity=$-NNP\frac{dP}{P} + URE\frac{dR}{R}$
  ⇒ $M\hat{P}C = 1; M\hat{P}S = 0$
  ⇒ dc & dn ∼ $-NNP\frac{dP}{P} + URE\frac{dR}{R}$
  ⇒ pure wealth effect
- When *unable to access financial market*
  ⇒ NNP=URE=0 (hand-to-mouth, $M\hat{P}C = 1$) ⇒ pure wealth effect

GE model

We proceed to discuss a general equilibrium model. In the GE model, we assume

- closed economy
- I heterogeneous types of agents $\{\beta_i, u_i, v_i\}$
  - each type has a mass 1 of individuals
- idiosyncratic state: $s_{it} \in S_i$
- idiosyncratic income change: dY_i
- gross income change: dY
- for any variable z, we denote $E_I[z_{it}]$ as cross sectional average.

Benchmark: RA model. *To the first order, in response to* $dY_i = dY$, dY, dP *and* dR, *aggregate consumption changes by*

$$dC = E_I[\frac{Y_i}{Y}M\hat{P}C_i]dY - E_I[\sigma_i(1 - M\hat{P}C_i)c_i]\frac{dR}{R}$$

or equivalently,

$$dC = \underbrace{M\hat{P}CdY}_{\text{Income channel}} - \underbrace{\sigma(1 - M\hat{P}C)C\frac{dR}{R}}_{\text{Substitution channel}}$$

$$\Rightarrow \quad \frac{dC}{C} = -\sigma\frac{dR}{R}$$

Theorem 3. *To the first order, in response to dY_i, dY, dP and dR, aggregate consumption changes by* dC=

$$\underbrace{E_I[\frac{Y_i}{Y}M\hat{P}C_i]dY}_{\text{Agr-income channel}} + \underbrace{Cov_I(M\hat{P}C_i, dY_i - Y_i\frac{dY}{Y})}_{\text{Earning hetero channel}} - \underbrace{Cov_I(M\hat{P}C_i, NNP_i)\frac{dP}{P}}_{\text{Fisher channel}}$$
$$+ (\underbrace{Cov_I(M\hat{P}C_i, URE_i)}_{\text{URE channel}} - \underbrace{E_I[\sigma_i(1 - M\hat{P}C_i)c_i]}_{\text{Substitution channel}})\frac{dR}{R} \quad (218)$$

⇒ Macroeconomic response captured by a small set of household-level micro data. (*sufficient statistics*)

⇒ Can be applied in mp, fp or even open economy analysis.

Question 1: Do re-distributional channels amplify mp shocks?
Rewrite equation as
dC=

$$E_I[\frac{Y_i}{Y}M\hat{P}C_i]dY + \gamma Cov_I(M\hat{P}C_i, \frac{Y_i}{Y})dY - Cov_I(M\hat{P}C_i, NNP_i)\frac{dP}{P}$$
$$+ (Cov_I(M\hat{P}C_i, URE_i) - E_I[\sigma_i(1 - M\hat{P}C_i)c_i])\frac{dR}{R} \quad (219)$$

where γ measures the elasticity of agent i's relative income to aggregate income. The effect of mp on γ is negative in literature.

Question 2: Do re-distributional channels amplify mp shocks?

⇔: Are the *Cov* terms positive or negative?

Answer: Negative. Re-distributional channels amplify mp shocks.

- Low-income agents have high MPCs.

$$Cov_I(M\hat{P}C_i, \frac{Y_i}{Y}) < 0$$

  MP accommodation ⇒ income inequality ↓ ⇒ Aggregate consumption ↑
  ⇒ MP accommodation *increases* aggregate consumption through income heterogeneity channel.

- Net nominal *borrowers* have higher MPC than Net nominal *lenders*.

$$Cov_I(M\hat{P}C_i, NNP_i) < 0$$

  MP accommodation ⇒ price ↑ ⇒ benefit borrowers ⇒ aggregate consumption ↑
  ⇒ MP accommodation *increases* aggregate consumption through Fisher channel.

- Agents with unhedged *borrowing* exposure ($URE < 0$) have higher MPC than agents with unhedged *saving* exposure ($URE > 0$).

$$Cov_I(M\hat{P}C_i, URE_i) < 0$$

  ⇒ MP accommodation *increases* aggregate consumption through interest rate exposure channel.

9. Firm Dynamics

9.1. *** Hopenhayn (1992, ECMA).** This paper is workhorse model of industry dynamics, which features endogenous stationary distribution with entry-and-exit, no aggregate uncertainty and frictionless economy (except a fixed operation cost).

Stylized facts on firm heterogeneity.

- size effect
  - size distribution of firms is skewed to the right and
  - the skewness of a cohort's size distribution declines with age
- investment
  - investment growth decreases with size and age, both unconditionally and conditionally
- employment
  - employment growth decreases with size and age, both unconditionally and conditionally
- entry and exit
  - exit hazard rate declines with age
  - entry rate is procyclical
  - exit rate is countercyclical

Settings.

- discrete and infinite time horizon
  - discount factor: β
- continuum of firms
  - law of large numbers holds
- homogeneous product
  - exogenous aggregate demand for output
  - single input: labor
  - exogenous aggregate supply of input
- entry and exit
  - potential entrants are *ex ante* identical
  - incumbents are heterogeneous in idiosyncratic productivity

Figure 29. Timeline

Firm's Problem.

- production technology:

$$f(a, n) = ay(n) = an^{\alpha}$$

  - a: idiosyncratic productivity, Markov process: a $\rightarrow$ a'
  - labor input
  - $\alpha < 1$: decreasing return to scale $\rightarrow$ optimal size
- role of fixed cost: c^f
  - generating endogenous exit
- operating profit:

$$\pi(a, p, w) = \max_n pf(a, n) - wn - c^f$$

  - optimal output denoted as $q^* := f(a, n^*)$
  - optimal input denoted as $n^* := n(a, p.w)$

Incumbent's Problem.

- two decisions:
  - size of employment: one-to-one mapping from productivity (a)
  - exit
- exit decision:
  - if exit: 0
  - if not exit: *expected* operating profit
- value function:

$$v_t(a; \mu) = \pi(a, p, w) + \beta max\{0, \int v_{t+1}(a'; \mu')F(da'|a)\}$$

  - μ : aggregate state (i.e., distribution, thus prices)
  - exit cut-off value a^*:

$$0 = \int v_{t+1}(a'; \mu')F(da'|a^*) \quad \text{or}$$
$$a^* = inf\{a \in A : \int v_{t+1}(a'; \mu')F(da'|a^*) \geq 0\}$$

Entrant's Problem.

- size of potential entrants: M_t
- one decision:
  - entry, after paying a sunk entry cost c^e
- entry decision:
  - enter if

$$\int v_t(a, \mu)g(da) \geq c^e$$

  - free entry:

$$\int v_t(a, \mu)g(da) = c^e \text{ if } M_t > 0$$

Distribution. Law of Motion:

$$\mu_{t+1}([0, a']) = \underbrace{\int_{a \geq a^*} F(a'|a)\mu_t(da)}_{\text{Continuing Incumbent}} + \underbrace{M_{t+1}G(a')}_{Entrants} \tag{220}$$

Define

$$\hat{P}_t = \begin{cases} \int_{a \in A} F(a'|a) \quad if \quad a \geq a^* \\ 0 \quad otherwise \end{cases}$$

$\Rightarrow$ Law of Motion:

$$\mu_{t+1} = \hat{P}_t\mu_t + M_{t+1}g \tag{221}$$

Equilibrium.

- aggregate supply (endogenous)

$$Q^s(\mu_t) = \int q_t(a, \mu)\mu_t(da)$$

- aggregate demand (exogenous)

$$Q^s$$

- aggregate labor demand (endogenous)
$$N^d(\mu_t) = \int n_t(a, \mu)\mu_t(da)$$
- aggregate labor supply (exogenous)
$$N^s$$
- both markets clear at equilibrium
- focus on stationary equilibrium
  - constant distribution over time

Distribution.

- Stationary Distribution:
$$\mu^* = \hat{P}\mu^* + M^* g \tag{222}$$
$$\Rightarrow \quad \mu^* = M^*(I - \hat{P})^{-1} g \tag{223}$$
  - stationary distribution is linearly homogeneous in m (scalar)
  - stationary distribution can be found by simulation as well. (appendix)

Comparative Statics.

- entry cost parameter: $c^e \uparrow$
  - expected discounted profits: $\uparrow$
  - exit threshold a^*: $\downarrow$
  - entrants mass m^*: $\downarrow$
  - output price p^*: $\uparrow$
  - entry rate/exit rate m^*/μ^*: $\downarrow$
  - firm-size distribution: ambiguous
    * price effect: incumbents increase output q^* and employment n^*
    * selection effect: more incumbent firms are relatively-low productivity firms
    * selection effect: entrants are of better productivity

Results.

- size effect
  - size of output $\leftrightarrow$ size of employment $\leftrightarrow$ productivity draw
  - unconditionally large firms have lower growth rate on average
- age effect
  - unconditionally old firms have lower growth rate on average
  - firms age as they survive in the market over time
  - no conditional age effect
- frictionless environment
  - model: young firms are small because they have lower draw on productivity
  - data: young firms are small not because they are inefficient
- Next Step: adding frictions to Hopenhayn (1992)

9.2. *** Cooley and Quadrini (2001, AER).** This paper incorporates *financial market friction* and *persistent idiosyncratic shock* to a workhorse model of firm dynamic to explain stylized facts introduced above.

Settings.

- persistent shock + financial constraint → size + age effect
  - conditional on age, the dynamics of firms are negatively related to the size of firms
  - conditional on size, the dynamics of firms are negatively related to the age of firms
- capture the features of the financial behavior of firm
  - small and younger firms pay fewer dividends, take on more debt, and invest more
  - small firms have higher values of Tobin's q
  - investment of small firms is more sensitive to cash flows
- financial frictions
  - equity: cost or premium associated with increasing equity
  - debt: costly default
  - trade-off theory

A Stylized Model.

- decreasing return to scale production technology:
$$y = af(k+b)$$
  - a: idiosyncratic productivity, i.i.d
  - k: owned capital (equity), no depreciation
  - b: borrowed capital (financed with debt)
- borrowing constraint:
$$b \leq k$$
  - interest rate: r
- value function:
$$v(k,b) = \max_{k',b'} af(k+b) - br - (k'-k) + \beta \int v(z',k')F(dz')$$
  - efficient size: $E\{af'(k^* + b^*)\} = r$
- optimal borrowing (constrained firm)
$$b' = k'$$
- capital accumulation:
$$k' = af(2k) - rk + k$$
- growth rate:
$$\frac{k'-k}{k} = \frac{af(2k)}{k} - r$$
  - decreasing in k
- financial constraint impedes firms to jump directly to their efficient size.

Full Model.

- assumptions
  - depreciation
  - inter-temporal debt

  - compound idiosyncratic shocks: persistent + transitory
  - financial market frictions

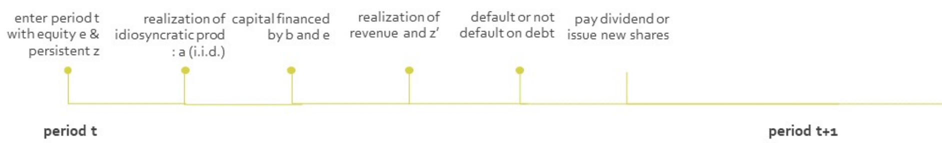


FIGURE 30. Timeline

Firm's problem.

- net worth end-of-period:
$$\pi(e, b, z + a) = (1 - \phi)(e + b) + (z + a)f(e + b) - (1 + \tilde{r})b$$
  - a: transitory productivity (accidents), i.i.d, unexpected
  - z: persistent productivity (technology), Markov process: z $\to$ z', revealed 1 period in advance
  - e: equity (asset)
  - ϕ: cost of capital (depreciation)
  - $\tilde{r}$: interest rate charged by intermediary
- endogenous default: threshold i.i.d shock $\underline{a}$ implicitly defined by
$$\underbrace{(1 - \phi)(e + b) + (z + \underline{a})f(e + b) - (1 + \tilde{r})b}_{=\pi(e,b,z+\underline{a})} = \underline{e}(z')$$
  - default if value of continuation is less than zero
  - threshold net worth of default: $\underline{e}(z')$
  - $e(z') < \underline{e}(z') \Rightarrow$ liability renegotiated until $e(z') = \underline{e}(z')$

External Finance: Debt.

- interest rate:
$$(1 + r)b = (1 + \tilde{r})b \int_{\underline{a}}^{\infty} g(da) + \int_{-\infty}^{\underline{a}} [(1 - \phi)(e + b) + (z + a)f(e + b) - \xi]g(da)$$
  - r: risk-free interest rate
  - ξ: default loss
- $\Rightarrow$ threshold i.i.d. shock $\underline{a} = \underline{a}(z, e, b, z')$:
$$(1 + r)b + \underline{e}(z') \int_{\underline{a}}^{\infty} g(da) + \xi \int_{-\infty}^{\underline{a}} g(da) = (1 - \phi)(e + b) + h(\underline{a})F(e + b) \qquad (224)$$
  - where $h(\underline{a}) = z + \underline{a} \int_{\underline{a}}^{\infty} g(da) + \int_{-\infty}^{\underline{a}} ag(da)$
- sequence of decisions: (a). default $\to$ (b). equity issuance/ dividend payment $\to$ (c). next period debt
- default does not lead to exit of the firm
- debt is re-negotiated after default

  - if $\pi(e,b,z+a) < \underline{e}(z')$, intermediary loss $= \underline{e}(z') - \pi(e,b,z+a)$
- end-of-period net worth

$$q(e,b,z+a,z') = \begin{cases} \underline{e} + (a-\underline{a})f(e+b), & if \quad a \geq \underline{a}(z,e,b,z') \\ \underline{e}, & if \quad a \leq \underline{a}(z,e,b,z') \end{cases} \tag{225}$$

External Finance: Equity.

- sequence of decisions: (a). default → (b). equity issuance/ dividend payment → (c). next period debt
- equity finance:

$$d(x,e') = \begin{cases} x - e', & if \quad x \geq e' \\ (x-e')(1+\lambda), & if \quad x \leq e' \end{cases} \tag{226}$$

  - where x: end-of-period equity of the firm before (b)
  - if $d(x,e')$ is positive, firm pays dividend;
  - if $d(x,e')$ is negative, firm issues equity, at cost λ;

Firm's Problem.

- sequence of decisions: (a). default → (b). equity issuance/ dividend payment → (c). next period debt
- value of the firm at the end of the period after (b) but before (c):

$$\Omega(z,e) = \max_b \{\beta \sum_{z'} \int_{\underline{a}} \tilde{\Omega}(z', q((e,b,z+a,z'))\Gamma(z'|z)f(da)\} \tag{227}$$

  s.t. equation (5) and (6)
  - where $\tilde{\Omega}(z,e)$: end-of-period value after (a) but before (b) s.t.

$$\tilde{\Omega}(z',\underline{e}) = 0$$

    s.t.

$$\tilde{\Omega}(z',x) = \max_{e'}\{d(x,e') + \Omega(z',e')\}$$

    s.t. equation (7)

Proposition 3. PROPOSITION 3: There exists a unique function $\Omega^*(z,e)$ that satisfies the functional equation (8). In addition, if for a_1 and a_2 sufficiently small, $g(a) < a_1$ for all $a < -a_2$, then

- the firm's solution is unique, and the policy rule b(z, e) is continuous in e;
- the input of capital k = e + b(z, e) is increasing in e;
- there exist functions $\underline{e}(z) < \hat{e}(z) < \bar{e}(z)$, $z \in Z$, for which the firm renegotiates the loan if the end-of-period resources are smaller than $\underline{e}(z)$, will issue new shares if they are smaller than $\hat{e}(z)$, and distribute dividends if they are bigger than $\bar{e}(z)$;
- the value function $\Omega^*(z,e)$, is strictly increasing and strictly concave in $[\underline{e}, \bar{e}]$.

Proposition 3: Comment. There exist functions $\underline{e}(z) < \hat{e}(z) < \bar{e}(z)$:

- if $e < \hat{e}(z)$: the firm issues new shares to increase equity level to $\hat{e}(z)$, as marginal increase in value w.r.t. $e > 1 + \lambda$
- if $\hat{e}(z) < e < \bar{e}(z)$: the firm will not issue new shares, as marginal increase in value w.r.t. $e < 1 + \lambda$
- if $\bar{e}(z) < e$: the firm distribute dividends,as marginal increase in value w.r.t. $e < 1$
- who issues equity?
  - with relatively lower net worth
  - with improvement in technology

Entrants.

- new firms are created with an initial value of equity raised by issuing new shares to an optimal size: $\hat{e}(z)$
- the cost of creating a new firm with initial productivity z:
$$\kappa + (1+\lambda)\hat{e}(z)$$
- surplus of entry:
$$\Omega(z, \hat{e}(z)) - \kappa - (1+\lambda)\hat{e}(z)$$
- free entry (general equilibrium property)
$$\Omega(z_N, \hat{e}(z_N)) = \kappa + (1+\lambda)\hat{e}(z_N)$$
- invariant measure of firms μ^* exists.

i.i.d shock: role of financial friction.

- z takes only two values: an absorbing shock $z_0 = 0$ and z_1
- conditional on surviving, the shock is i.i.d.
- isolate the financial mechanisms from persistence mechanism
- key properties of the financial behavior of firm
  - small firms take on more debt (higher leverage).
  - small firms face higher probability of default.
  - small firms have higher rates of profits.
  - small firms issue more shares and pay fewer dividends.
- key properties of firm dynamics
  - small firms grow faster and experience higher volatility of growth.
  - small firms face higher probability of default.
  - small firms experience higher rates of job reallocation.
  - *without conditioning on size*, young firms experience higher rates of growth, default, and job reallocation.

persistent shock: interaction with financial friction.

- conditional on surviving, z follows a symmetric two-state Markov process
- firms differ over two dimensions: equity and productivity
- conditional on equity size, high productivity firms borrow more and implement larger production scales. trade-off:
  - a larger production scale allows higher expected profits
  - a larger production scale implies higher volatility of profits
- size dependence and age dependence in the dynamics of firm
  - unconditionally and conditionally
  - size dependence as before
  - age dependence derives from heterogeneous composition of firm types in each age class of firms
  - effect of entry: initial productivity of new firms
  - effect of persistent shock
  - age effect is more important for small firms; and it almost disappears for very large firms.

9.3. *** Arellano, Bai and Zhang (2012, JME).** This paper uses a firm dynamic model with *financial market friction* and *persistent idiosyncratic shock* to explain different stylized facts between developing economy and developed economy.

Introduction.

- firm dynamic: size effects
  - size-growth relation: size $\uparrow \Rightarrow$ growth $\downarrow$
  - size-leverage relation: size $\uparrow \Rightarrow$ leverage $\downarrow$
  - frictionless economy: no size effects
  - theory: financial friction [27]; adjustment cost; trade etc.
- effects *conditional* on
  - firm characteristics: age, sector etc.
  - U.S. economy: industry structure, financial development etc
- this paper: condition of financial development $\Rightarrow$ size effects
  - cross-country variation
  - financial development $\leftrightarrow$ size-growth, size-leverage
  - quantitative model

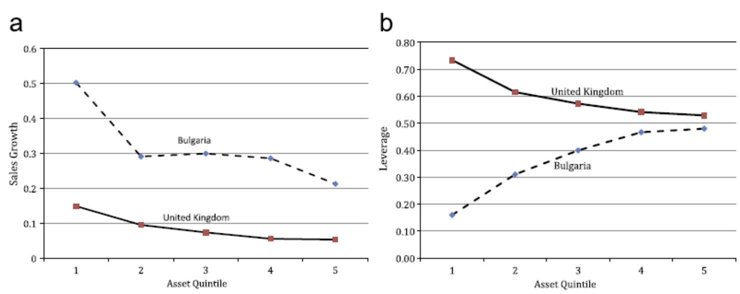


Fig. 1. Firm size, leverage and sales growth. (a) Size and growth. (b) Size and leverage.

Empirical.

Stylized Fact 1.

- size-growth relation (panel a)
  - small firms grow faster than large firms
  - difference is larger in Bulgaria
- size-leverage relation (panel b)
  - Bulgaria: small firms use less debt financing
  - UK: small firms use more debt financing

Stylized Fact 2.

- database: Amadeus
  - 27 European countries
  - 2.6 million firms in non-financial, non-public sectors
- regression:

$$y_{k,c} = \beta_0 + \beta_1 size_{k,c} + \beta_2 size_{k,c} * FD_c + Dummy + v_{k,c} \tag{228}$$

- dependent variables ($y_{k,c}$): growth, leverage

[27] Cooley and Quadrini (2001), Albuquerque and Hopenhayn (2004), Clementi and Hopenhayn (2006), and DeMarzo and Fishman (2007) etc.

  - growth = growth rates of sales
  - leverage = total debt / total asset
- independent variables: $size, FD, dummy$
  - size: book value of the firm's total asset
  - FD: development of financial markets
    - ∗ average private credit to GDP ratio (+)
    - ∗ share of banks' overhead costs in total bank assets (-)
    - ∗ coverage of credit bureaus (+)
  - dummy: fixed effects of country, industry and age

Table 2
Firm leverage, growth and financial development.

	Leverage			Sales growth		
	(1)	(2)	(3)	(1)	(2)	(3)
Size	0.021***	0.014***	0.018***	−0.134***	0.024***	−0.082***
	(0.0002)	(0.0003)	(0.0001)	(0.0016)	(0.0011)	(0.0010)
FD × *Size*	−0.006***	0.050***	−0.005***	0.097***	−1.880***	0.051***
	(0.0002)	(0.0048)	(0.0002)	(0.0013)	(0.0310)	(0.0008)
Adjusted R^2	0.28	0.27	0.28	0.06	0.06	0.06
Observations	2 621 201	2 606 324	2 621 201	2 621 201	2 606 324	2 621 201

Notes: *Size* is measured by the logged asset share of a firm. *FD* denotes financial development, measured by private credit to GDP (1), overhead costs (2) or credit bureau coverage (3). All regressions have a fixed effect at the country × industry × age level. The standard errors reported in parentheses are robust to heteroskedasticity. *** denotes significant at 1%.

implied y-size coefficient = $\beta_1 + \beta_2 * FD_c$

Country	FD(1)	size-leverage	size-growth
UK	1.42	0.012	0.004
Germany	1.16	0.014	-0.021
Sweden	0.89	0.016	-0.048
Median	*0.47*	*0.018*	*-0.088*
Bulgaria	0.22	0.020	-0.113

- size–leverage relation
  - median financial market: size ↑ → leverage ↑
  - financial development ↑ ⇒ size-leverage slope ↓
- size–growth relation
  - median financial market: size ↑ ⇒ growth ↓
  - financial development ↑ ⇒ size-growth slope ↑
- financial development and size effects
  - FD ↑ ⇒ size effects ↓: small firm ∼ large firm
  - FD ↑ ⇒ 'distortion' ↓ for small firms

Full Model. Decreasing return to scale technology:

$$y = zK^{\alpha}, \quad 0 < \alpha < 1 \tag{229}$$

- z: idiosyncratic prod
  - z: Markov process, $f(z', z)$
  - $\log(z) = \log(\mu) + \log(\varepsilon)$
  - permanent component (*productivity*): $\{\mu_z^i, i = 1 : 5\}$

  - stochastic component (*luck*): $\{\varepsilon_l, \varepsilon_h\}$
  - θ: prob of exogenous death
- K: capital stock
  - depreciation: δ
  - net investment: $K' - (1-\delta)K$
  - adjustment cost: $\phi(K'-K)^2/K$
  - degree of friction: ϕ

Debt Contract.

- debt contract:

$$(B', B'_R) \in \Omega(K', z) \tag{230}$$

  B': new loan. B'_R: face value.
- recovery value if firms default:

$$R(K') = max\{(1-\psi)(1-\delta)K' - \phi K', 0\} \tag{231}$$

- break-even condition

$$B' + \xi = \frac{B_R(1 - \int \tilde{d} f(z', z)dz') + R(K') \int \tilde{d} f(z', z)dz'}{1+r} \tag{232}$$

- parameters
  - recovery rate: $1-\psi$
  - financial intermediation cost: ξ (proxy for financial development)
  - binary default decision: $\tilde{d} = d(K, B_R, z)$

Equity.

- dividend:

$$D = zK^\alpha - B_R + B` - K' + (1-\delta)K - \phi(K'-K)^2/K \tag{233}$$

- value function:

$$V(K, B_R, z) = \max_{\tilde{d}\in\{0,1\}} (1-\tilde{d})V^c(K, B_R, z) \tag{234}$$

- value function conditional on repayment:

$$V^c(K, B_R, z) = \max_{D, K', (B', B'_R)\in\Omega} (1 + \gamma 1_{D<0})D + \beta E_z V(K', B'_R, z') \tag{235}$$

Entrants.

- entrant:

$$V^e(K_0, 0, z) = \max_{D, K', (B', B'_R)} (1 + \gamma_e 1_{D<0})D + \beta E[V(K', B'_R, z')] \tag{236}$$

  subject to

$$D = B' - K' - \phi(K' - K_0)^2/K_0 \tag{237}$$

  and $z' \sim g(z')$
- mass of project = 1
  - project: exit firms → potential entrants

Distribution.

- distribution: $s \equiv (K, B_R, z)$

$$\begin{aligned}\Gamma(s') = &\int [1-d(s)]Q(s,s')f(z',z)\Gamma(s)d(K\times B_R \times z)\\ &+\int d(s)Q_e(s')g(z')\Gamma(s)d(K\times B_R\times z)\end{aligned} \tag{238}$$

where transition functions are:

$$Q(s',s) = \begin{cases} 1, & if \quad K'(K,B_R,Z)=K', B_R'(K,B_R,Z)=B_R' \\ 0, & otherwise \end{cases} \tag{239}$$

and for entrants

$$Q_e(s') = \begin{cases} 1, & if \quad K'(K_0,0)=K', B_R'(K_0,0)=B_R' \\ 0, & otherwise \end{cases} \tag{240}$$

An Aside: Analytical Solution.

- assumptions
  - idiosyncratic prod shock (permanent and transitory)
  - capital adjustment cost and partial full depreciation
  - equity financing: proportional cost
  - debt financing: default risk with partial no recovery
  - debt creditor: fixed cost (*proxy for FD*)
- value function conditional on repayment:

$$V^c(K,B_R,z) = \max_{K',B'} zK^\alpha - B_R + B' - K' + \beta V(K', B_R', z) \tag{241}$$

- assumption: $\beta(1+r)<1$ and ξ sufficiently small:

$$K' = K_{fb}(z) : z\alpha K_{fb}^{\alpha-1} = 1+r \tag{242}$$

- debt limit and repayment denoted as $\bar{B}(z)$ and $\bar{B}_R(z)$

$$\bar{B}(z)+\xi = \frac{\bar{B}_R(z)}{1+r} \tag{243}$$

- value function conditional on repayment:

$$V^c(K_{fb},\bar{B}_R,z) = zK_{fb}^\alpha - \bar{B}_R + \bar{B} - K_{fb} + \beta V(K_{fb},\bar{B}_R,z) \tag{244}$$

- no default at debt limit: $V(K_{fb},\bar{B}_R,z) = V^c(K_{fb},\bar{B}_R,z)$

$$V^c(K_{fb},B_R,z) = [zK_{fb}^\alpha - K_{fb} - r\bar{B}(z) - (1+r)\xi]/(1-\beta) \tag{245}$$

- debt limit derived from:

$$V^c(K_{fb},B_R,z)=0 \tag{246}$$

- debt limit:

$$\bar{B}(z) = \frac{(1+r-\alpha)}{r\alpha}K_{fb}(z) - \frac{1+r}{r}\xi \tag{247}$$

- leverage ratio:

$$\frac{\bar{B}(z)}{K_{fb}(z)} = \frac{(1+r-\alpha)}{r\alpha} - \frac{1+r}{r}\frac{\xi}{K_{fb}(z)} \tag{248}$$

- size-leverage relation
  - larger firm $\leftrightarrow$ higher leverage

  - fixed credit cost ξ affects small firm disproportionately
- fixed credit cost $\to$ size-leverage relation
  - $\xi = 0$: no size effect on leverage
  - $\xi \uparrow$: size effect on leverage $\uparrow$

Table 6
Benchmark parameters and target moments.

Calibrated parameters		
Discount factor	β	0.96
Interest rate	r	0.04
Capital depreciation rate	δ	0.10
Technology	α	0.65
Equity issuance cost	γ	0.30
Capital loss after default	ψ	0.25
Death rate	θ	0.072
Shock persistence	ρ	0.86
Estimated parameters		
Permanent productivity	c	0.550
Stochastic shock variance	σ	0.525
Capital adjustment cost	ϕ	0.001
Credit cost	ξ	0.010
Entrant starting capital	K_0	0.002
Entrant equity issuance cost	γ_e	0.130

Calibration.

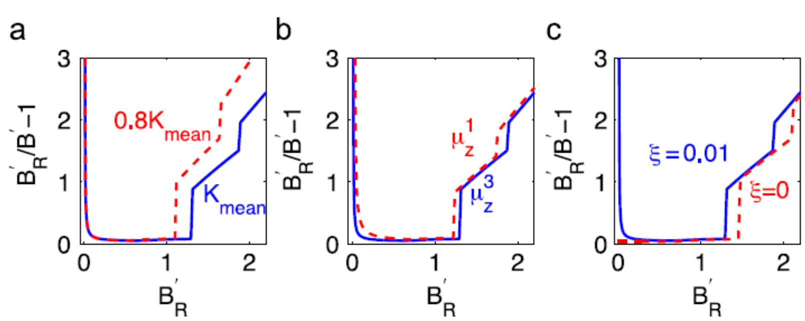


FIGURE 31. Sensitivity of Debt Schedule

Quantitative Analysis.

- debt contract: $(B', B'_R) \in \Omega(K', z)$
  - effective interest rate (spread) $= \frac{B'_R}{B'} - 1$
  - spread in U-shape
  - high for small loans: fixed credit cost ξ
  - high for large loans: default risk
- sensitivity of debt contract (figure)
  - sensitivity to K': collateral effect (panel a)

- sensitivity to μ (panel b)
- sensitivity to ξ (panel c)

- stochastic productivity process: quantitative exploration
  - median permanent shock ($\mu = \mu_z^3$)
  - low stochastic shock ($\varepsilon = \varepsilon_l$)
  - average capital stock $K = K_{mean}$ with median productivity
- policy rule: $K'(K, B_R, z), D(K, B_R, z), B'(K, B_R, z)$
  - median permanent shock ($\mu = \mu_z^3$)
  - low stochastic shock ($\varepsilon = \varepsilon_l$)
  - average debt level $B = 0.43 * K_{mean}$

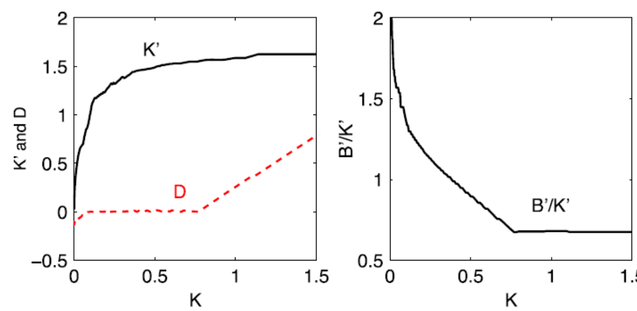


Fig. 3. Policy rules. *Note*: This figure plots the optimal capital choice K', dividends D, and the ratio of the loan choice relative to the capital choice B'/K' as a function of the beginning capital K for a firm with median permanent productivity μ_z^3, stochastic shock ϵ_l and debt at 43% of the average capital across the μ_z^3-firms. All values on the axis are relative to the average capital across the μ_z^3-firms.

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FIGURE 32. Policy Rule

- policy rules: size effect (figure)
  - smallest firm [0%-20%]
  - medium firm [20%-75%]
  - largest firm [75%-]

Table 7
Quantitative model results.

	Bulgaria data		Bulgaria benchmark		Zero credit cost	
	Growth	Leverage	Growth	Leverage	Growth	Leverage
All firms						
Mean	0.32	0.36	0.34	0.48	0.30	0.68
Small firms	0.37	0.26	0.62	0.32	0.34	0.65
Large firms	0.26	0.46	0.05	0.64	0.26	0.71
Difference	0.11	−0.20	0.57	−0.32	0.08	−0.06

FIGURE 33. Model Moments

Model Moments.

- leverage: unproductive vs unlucky
  - unproductive: low permanent shock → high spread → lower leverage
  - unlucky: sequence of low transitory shock → higher leverage
- growth
  - hit by good transitory shock → higher growth → efficient level
- counterfactual: credit cost (ξ)
  - inefficiency: unfavorable debt schedule for small firms

Robustness.

- Regression 1:
$$Growth_k = \beta_0 + \beta_1 size_k + e_k$$
  - $\beta_1 < 0$: size-growth relation
- Regression 2:
$$Leverage_k = \beta_0 + \beta_1 size_k + e_k$$
  - $\beta_1 > 0$: size-leverage relation
- Regression 3:
$$y_{k,c} = \beta_0 + \beta_1 size_{k,c} + \beta_2 size_{k,c} * (Credit/GDP)_c + e_{k,c}$$
  - y: zero-leverage dummy = 1 if leverage is zero.
  - $\beta_1 > 0$: size-leverage relation
  - $\beta_2 < 0$: financial development $\rightarrow$ size-leverage relation

Conclusion.

- benchmark size effects
  - small firms grow faster than large firms
  - small firm use less debt financing than large firms
- as financial development improves
  - growth rate of small firms relative to large firm decreases
  - leverage ratio of small firms relative to large firm increases
- micro-data into macro quantitative model
  - growth and financing patterns
  - across firms and across country

9.4. *** Khan and Thomas (2008, ECMA).** The model of Khan and Thomas (2008) is a workhorse *general equilibrium* models with heterogeneous firms. In the model, the distribution of firms over idiosyncratic states $\{\varepsilon, k\}$ has non-trivial role in shaping aggregate economy.

Household. There is a representative household whose preferences are represented by the utility function

$$\max E_t \sum_{t=0}^{\infty} \beta^t [\frac{C_t^{1-\sigma} - 1}{1-\sigma} - \chi \frac{N_t^{1+\alpha}}{1+\alpha}]$$

The household owns all the firms in the economy and markets are complete.

Firms.

- DRS technology using capital and labor as input

$$y_{jt} = e^{z_t} e^{\varepsilon_{jt}} k_{jt}^{\theta} n_{jt}^{\nu} \tag{249}$$

  where z_t is aggregate productivity shock and ε_{jt} is idiosyncratic shock, both of which follow AR(1). $\theta + \nu < 1$.
- fixed adjustment cost of capital:

$$k_{jt+1} = (1-\delta)k_{jt} + i_{jt}$$

  where i_{jt} is new investment. If $\frac{i_{jt}}{k_{jt}} < -a$ or $\frac{i_{jt}}{k_{jt}} > a$, the firm must pay additional ξ_{jt} unit of labor, which is a stochastic and i.i.d from a uniform distribution over [0, $\bar{\xi}$].
- Bellman equation of firms

$$v(\varepsilon, k, \xi; s) = \lambda(s) max_n \{y_{jt} - w(s)n\} + \max\{v^a(\varepsilon, k, \xi) - \xi\lambda(s)w(s), v^n(\varepsilon, k, \xi)\} \tag{250}$$

  where marginal utility of consumption $\lambda(s) = C(s)^{-\sigma}$, $v^a(\varepsilon, k, \xi)$ denote value if a firm pays adjustment cost and invest:

$$v^a(\varepsilon, k, \xi) = \max_{k' \in R} \lambda(s)[(1-\delta)k - k'] + \beta E[v(\varepsilon', k'; \hat{s'}|\varepsilon, k; s)] \tag{251}$$

  where $v(\varepsilon, \hat{k}; s) = \int(\varepsilon', k'; s'|\varepsilon, k; s)dG(\xi)$ and $v^n(\varepsilon, k, \xi)$ denote value if a firm invest within the rate [-a,a]:

$$v^n(\varepsilon, k, \xi) = \max_{k' \in [(1-\delta-a)k, (1-\delta+a)k]} \lambda(s)[(1-\delta)k - k'] + \beta E[\int v(\varepsilon', k'; s'|\varepsilon, k; s)dG(\xi)] \tag{252}$$

- unique threshold value of fixed cost $\xi(\varepsilon, k)$ making the firm indifferent between unconstrained or constrained investment:

$$\xi(\varepsilon, \tilde{k}; s) = \frac{v^a - v^n}{\lambda(s)w(s)} \tag{253}$$

- the expectation over ξ can be expressed analytically:

$$v(\varepsilon, \hat{k}; s) = \lambda(s) \max_n \{y_{jt} - w(s)n\} + \frac{\hat{\xi}}{\bar{\xi}}[v^a(\varepsilon, k; s) - \lambda(s)w(s)\frac{\xi(\varepsilon, \hat{k}, \xi)}{2}] + (1 - \frac{\hat{\xi}}{\bar{\xi}})v^n(\varepsilon, k; s) \tag{254}$$

Equilibrium.

- firm: solve problem above;
- household: $\lambda(s) = C(s)^{-\sigma}$
- output market clearing

$$C(s) = \int [y + (1-\delta)k - \frac{\hat{\xi}}{\bar{\xi}} k^a(\varepsilon, k; s) + (1 - \frac{\hat{\xi}}{\bar{\xi}}) k^n(\varepsilon, k; s)]$$

- labor market clearing

$$\int [n(\varepsilon, k; s) + \frac{\hat{\xi}^2(\varepsilon, k; s)}{2\bar{\xi}}] g(\varepsilon, k) d\varepsilon dk = [\frac{w(s)\lambda(s)}{\chi}]^{\frac{1}{\alpha}}$$

- law of motion for distribution

$$g'(\varepsilon', k'; s') = \int \int [\frac{\hat{\xi}}{\bar{\xi}} \{k^a(\varepsilon, k; s) = k'\} + (1 - \frac{\hat{\xi}}{\bar{\xi}}) \{k^n(\varepsilon, k; s) = k'\}] f(\varepsilon' | \varepsilon) d\varepsilon dk \tag{255}$$

- law of motion for aggregate productivity

$$z' = \rho_z z + \sigma_z \omega_z' \tag{256}$$

Results.

- match cross-sectional investment distribution
  - lumpy investment
  - persistence capital and productivity heterogeneity
  - GE effect important for third or fourth moments.
- undesired feature I: cyclicality of interest rate
  - this paper: procyclical
  - data: countercyclical
  - Winberry (2020) solved this issue with internal habit and quadratic adjustment costs
- undesired feature II: firm's life cycle
  - this paper: no entry-exit; no role for age
  - data: entry-exit and young firm matters for aggregate dynamics
  - Khan and Thomas (2013), Clementi and Palazzo (2016) explore this dimension

9.5. *** Khan and Thomas (2013, JPE).** Khan and Thomas (2013) is a classic paper with

- macro model studying credit shock
  - as disturbance to asset collateral value (Jermann & Quadrini 12')
  - with rich firm heterogeneity
  - qualitatively different recession from tfp-driven ones
- firm dynamic model with
  - real and financial frictions
  - inefficient capital allocation
  - non-trivial macroeconomic effects
- first DSGE model combining
  - firm heterogeneity
  - real frictions
  - financial frictions (Kiyotaki & Moore 97')
- numerical method of independent merit

Failure of Neoclassical Investment Model.

- A standard neoclassical firm's problem:

$$\begin{aligned} max \quad & k_{it}^{\alpha} - i_{i,t} - \frac{1}{2}\phi(i_{it}/k_{it})^2 k_{it} + \frac{1}{1+r}v(k_{it+1}) \\ s.t. \quad & k_{it+1}^{\alpha} = (1-\delta)k_{it} + i_{i,t} \quad (multiplier : q_{it}) \end{aligned}$$

  f.o.cs

$$q_{it} = v'(k_{it+1})$$
$$q_{it} = 1 + \phi(i_{it}/k_{it})$$

- Two implications of the q-theory model:
  1. q_{it} is the marginal value of capital to the firm;
  2. investment (ratio) is positively related to q_{it}:

$$i_{it}/k_{it} = \phi^{-1}(q_{it} - 1)$$

- Proxy for q (under constant returns):

$$v'(k_{it}) = \frac{v(k_{it})}{k_{it}}$$

$$q_{it} = \frac{v(k_{it+1})}{k_{it+1}} = \frac{1}{1+r}\sum_{s}(\frac{1-\delta}{1+r})^s[\alpha k_{it+s+1}^{\alpha-1} + \phi_{it+s+1}]$$

- Empirical regression:

$$\frac{i_{it}}{k_{it}} = \alpha_i + \beta q_{it} + \mathbf{B}ctrvar_{it} + \varepsilon_{it}$$

- Failures of neoclassical investment model:
  - Coefficient β is estimated to be small and unstable;
  - Coefficients on $ctrvar$s, especially cash flow, are large and significant.
- Lessons from failures of neoclassical investment model:
  - Real frictions (non-convex adjustment costs etc.) are important;
  - Financial frictions (borrowing constraints etc.) are important.

Why adding frictions?

- Frictionless economy (two-period model):

$$\max_{k_{i1},b_{i1}} d_{i0} + \frac{1}{R}E[d_{i1}]$$

$$d_{i0} = x_{i0} + \frac{1}{R}b_{i1} - k_{i1}$$

$$d_{i1} = z_{i1}k_{i1}^{\alpha} - b_{i1}$$

  solution (*MM theorem*):

$$k_{i1} = (\frac{\alpha E[z_{i1}]}{R})^{\frac{1}{1-\alpha}}$$

  $\rightarrow$ any finite b and d optimal
  $\Rightarrow$ frictionless model makes no prediction about financial variables
- Financial friction:
  - common frictions to equity finance:
    - cannot raise new equity: $d_{i0} \geq 0$
    - costly to raise new equity: pay some cost if $d_{i0} < 0$
    - dividend adjustment cost: $\phi(d_{i0}, d^*)$
  - common frictions to debt finance:
    - collateral constraint: $b_{i0} \leq$ (some measurement of) collateral value
    - limited commitment: default risk $\rightarrow$ risk premium

    $\Rightarrow$ non-trivial effects of financial variables for investment!
  - Frictions in this paper:
    a. (equity) cannot raise new equity: $d_{i0} \geq 0$
    b. (debt) collateral constraint: $b_{i0} \leq$ collateral value
- Firm heterogeneity:
  - k: predetermined capital
    - some degree of specificity
    - partial investment irreversibility
    - when $i > 0$, $k' = (1-\delta)k + i$
      when $i < 0$, $\theta_k k' = \theta_k(1-\delta)k + i$, $\theta_k < 1$
  - b: constrained borrowing
    - current capital as collateral
    - taken specificity into account
    - borrowing constraint

$$b' \leq \zeta_l \theta_k k$$

  - ε: idiosyncratic productivity
    - production function

$$y = z\varepsilon F(k, n)$$

    - persistent shocks to z
    - persistent shocks to ε
- frictions[29] + heterogeneity:
  - (real) partial irreversibility:
    - lumpiness: frequency of large investment
    - persistence: positive auto-corr of investment

[29] There is no frictions in labor market: so that same $(k, \varepsilon) \rightarrow$ same (n,y).

    * investment rules of (S,s) type
  - (real) partial irreversibility+ idiosyncratic shocks:
    large but unproductive firms cannot adjust to optimal level
  - (financial) borrowing constraint + idiosyncratic shocks:
    small but productive firms cannot adjust to optimal level
    $\Rightarrow$ disproportionate capital stock to productivity.
  - Does such misallocation amplify credit shock?

Model. Firms.

- Expected value *before* the beginning of each period:

$$v_0(k,b,\varepsilon;s,\mu) = (1-\pi_d)v(k,b,\varepsilon;s,\mu) + \pi_d \max_n [z\varepsilon F(k,n) - \omega(s,\mu)n + \theta_k(1-\delta)k - b] \tag{257}$$

- Value of continuation at the beginning of each period:

$$v(k,b,\varepsilon;s,\mu) = \max\{v^u(k,b,\varepsilon;s,\mu), v^d(k,b,\varepsilon;s,\mu)\} \tag{258}$$

- Upward Adjusting Firm:

$$v^u(k,b,\varepsilon;s,\mu) = \max_{n,k',b',D}[D + E_{s'}d_{s'}E_{\varepsilon'}v_0(k',b',\varepsilon';s',\mu')] \tag{259}$$

  s.t.

$$k' \geq (1-\delta)k$$
$$b' \leq \zeta_l \theta_k k$$
$$D = z\varepsilon F(k,n) - \omega(s,\mu)n + q(s,\mu)b' - b - [k' - (1-\delta)k] \geq 0$$
$$\mu' = \Gamma(s,\mu)$$

- Downward Adjusting Firm:

$$v^d(k,b,\varepsilon;s,\mu) = \max_{n,k',b',D}[D + E_{s'}d_{s'}E_{\varepsilon'}v_0(k',b',\varepsilon';s',\mu')] \tag{260}$$

  s.t.

$$k' \leq (1-\delta)k$$
$$b' \leq \zeta_l \theta_k k$$
$$D = z\varepsilon F(k,n) - \omega(s,\mu)n + q(s,\mu)b' - b - \theta_k[k' - (1-\delta)k] \geq 0$$
$$\mu' = \Gamma(s,\mu)$$

Household.

- preference

$$V^h(\lambda,\phi;s,\mu) = \max_{c,n^h,\phi',\lambda'}[U(c,1-n^h) + \beta E_{s'}V^h(\lambda',\phi';s',\mu')] \tag{261}$$

- budget constraint

$$c + q\phi' + \int_S \rho_1 \lambda'(d[k' \times b' \times \varepsilon']) \leq [\omega n^h + \phi + \int_S \rho_0 \lambda(d[k \times b \times \varepsilon])]$$
$$\mu' = \Gamma(s,\mu)$$

  where: current share holding: λ, value of current share: ρ_0;
  where: matured bond: ϕ;
  where: future share holding: λ', value of current share: ρ_1;
  where: future bond: ϕ', bond price: $1/q$.
  $\Rightarrow C^h(\lambda,\phi;s,\mu)$; $N^h(\lambda,\phi;s,\mu)$; $\Phi^h(\lambda,\phi;s,\mu)$; $\Lambda^h(k',b',\varepsilon';\lambda,\phi;s,\mu)$

Recursive Equilibrium.

$$\Lambda^h(k', b', \varepsilon'; \lambda, \phi; s, \mu) = \mu'(k', b', \varepsilon'; s, \mu)$$

$$N^h(\lambda, \phi; s, \mu) = \int_S [N(k, \varepsilon; s, \mu)]\mu(d[k \times b \times \varepsilon])$$

$$C^h(\lambda, \phi; s, \mu) = \int_S [y - (1 - \pi_d)IC + \pi_d(\theta_k(1-\delta)k - k_0)]\mu(d[k \times b \times \varepsilon])$$

$$\Phi^h(\lambda, \phi; s, \mu) = \int_S [B(k, b, \varepsilon; s, \mu)]\mu(d[k \times b \times \varepsilon])$$

Numerical Method.

- Outline:
  - Subsume household's problem into the firm's problem
    - replacing prices of labor, bond, output and discount factors
  - Solve firm's decision rules on dividend, capital and debt
    - sorting firms to two types: constrained and unconstrained
    - constrained firms exposed to binding borrowing constraint
    - unconstrained firms permanently free from borrowing constrained
  - Krusell-Smith algorithm to solve the problem numerically
    - nonlinear, iterative and computationally intensive
    - we do have better algorithm now
- Step 1: Subsume household's problem into the firm's problem
  - output price[30]:

$$p(s, \mu) = D_1U(C, 1 - N) \tag{262}$$

  - real wage: = MRS(c,n)

$$\omega(s, \mu) = \frac{D_2U(C, 1 - N)}{D_1U(C, 1 - N)} = \frac{D_2U(C, 1 - N)}{p(s, \mu)} \tag{263}$$

  - bond price: = expected gross real interest rate

$$q(s, \mu) = \frac{\beta E_s D_1U(C', 1 - N')}{D_1U(C, 1 - N)} = \frac{\beta E_s D_1U(C', 1 - N')}{p(s, \mu)} \tag{264}$$

  - firm's discount factor: consistent with MRSc,n

$$d(s, \mu) = \beta D_1U(C', 1 - N')/D_1U(C, 1 - N)$$

- Step 2: Reformulate firm's problem
  - Expected value *before* the beginning of each period[31]:

$$\begin{aligned} V_0(k, b, \varepsilon; s, \mu) = (1 - \pi_d)V(k, b, \varepsilon; s, \mu) + \pi_d \max_n p(s, \mu) \\ \times [z\varepsilon F(k, n) - \omega(s, \mu)n + \theta_k(1 - \delta)k - b] \end{aligned} \tag{265}$$

---

[30]We implicitly assume that firms discount by the same factor as households

[31]J(x)=1 if $x \geq 0$; $J(x) = \theta_k$ if $x < 0$;
and we exploit the fact that labor decision is static, independent of k' and b'.

– Expected value *at* the beginning of each period:

$$V(k, b, \varepsilon; s, \mu) = \max_{n,k',b',D} [p(s, \mu)D + \beta E_{s'} E_{\varepsilon'} V_0(k', b', \varepsilon'; s', \mu')] \tag{266}$$

s.t.

$$D \geq 0$$

$$z\varepsilon F(k, n) - \omega n + qb' - b - J(k' - [1 - \delta]k)[k' - (1 - \delta)k] - D \geq 0 \tag{267}$$

$$\zeta_l \theta_k k - b' \geq 0 \tag{268}$$

– Firms solve eq(9)-(12), taken $\{p, \omega, q\}$ as given
– Static labor choice:

$$z\varepsilon D_2 F(k, n^*) = \omega$$

– Profit:

$$\pi(k, b, \varepsilon; s, \mu) = z\varepsilon F(k, n^*) - \omega n^* - b \tag{269}$$

– Determination of $[D, k', b']$
  * most challenging objects
  * sort firms into two types
  * constrained firms: D=0 ↔ k' → b'
  * unconstrained firms: k' unaffected by borrowing limits

- Unconstrained Firms
  – Multiplier on borrowing constraints are zero
    → sufficient capital to circumvent collateral constraint
    → capital choice independent of financial position
  – Indifferent b/w saving and dividends[32]
    → indifferent about b'
    → mv of firm's retained earning (saving) = household (p)
  – b affecting value only through profit $\pi(k, b, \varepsilon; s, \mu)$

$$W(k, b, \varepsilon) = W(k, 0, \varepsilon) - pb$$

  – Target capital stocks (k^*)

$$k_u^*(\varepsilon) = arg \max_{k'} [-pk' + \beta E_{s'} E_{\varepsilon'} W_0(k', \varepsilon'; s', \mu')] \tag{270}$$

$$k_d^*(\varepsilon) = arg \max_{k'} [-p\theta_k k' + \beta E_{s'} E_{\varepsilon'} W_0(k', \varepsilon'; s', \mu')] \tag{271}$$

  – Capital decision rule: (S, s) form

$$K^w(k, \varepsilon; s, \mu) = \begin{cases} k_u^*(\varepsilon; s, \mu), & \text{if } k_u^* > (1 - \delta)k \\ (1 - \delta)k, & \text{if } k_u^* < (1 - \delta)k < k_d^* \\ k_d^*(\varepsilon; s, \mu), & \text{if } k_d^* < (1 - \delta)k \end{cases} \tag{272}$$

  – $D^w(k, b, \varepsilon; s, \mu)$ is implied given the decision rule for k and b.

- Constrained Firms
  – Value function of constraint firm:

$$V^c(k, b, \varepsilon; s, \mu) = max\{V^u(k, b, \varepsilon; s, \mu), V^d(k, b, \varepsilon; s, \mu)\} \tag{273}$$

  – Given (k, ε), find a cut-off debt level where
    * non-negative investment is possible

[32]We have to impose additional assumptions on saving policy rule (minimum saving) to guarantee so in all future dates and states.

    * borrowing constraint is not violated
    * avoid negative dividends
  - max b with $k' = (1-\delta)k$ and $D \geq 0$:

$$\hat{b} = q\zeta\theta_k k + z\varepsilon F(k, n^*) - \omega n^*$$

  - $b > \hat{b} \to$ downward adjustment: $V^d(k, b, \varepsilon; s, \mu)$
  - $b < \hat{b} \to$ upward adjustment: $V^u(k, b, \varepsilon; s, \mu)$
- Distinction b/w Unconstrained and Constrained Firms
  - If a firm can:
    * adopt capital rule of unconstrained firm
    * hold debt level within saving function
    * pay non-negative dividend
  - The firm is indistinguishable from unconstrained firm with (k, ε)

$$\begin{aligned} V(k, b, \varepsilon; s, \mu) &= W(k, b, \varepsilon; s, \mu) \quad , iff \quad D^w(k, b, \varepsilon; s, \mu) \geq 0 \\ &= V^c(k, b, \varepsilon; s, \mu) \quad , otherwise \end{aligned} \tag{274}$$

- Step 3: Solve the Problem (K-S algorithm)
  - Computational challenges
    * presence of investment irreversibility
    * collateral constraint
    * firm level productivity shocks
  - Curse of dimensionality:
    * individual state variable: $\{k, b, \varepsilon\}$
    * necessity to track their joint distribution: μ
    * aggregate state variable: $\{s, \mu\} = \{z, \zeta; \mu\}$
    * high-dimensional object
  - Approximation of aggregate state
    * $\{s, \mu\} \to \{s, m, \nu_1, \nu_2\}$
    * m: unconditional mean of capital
    * ν_1, ν_2: lagged indicators of credit crisis

  In each iteration,
  - solve value function in an inner loop
    * m' and p taken as given
    * interpolation of functions at knots of individual and aggregate states
    * piece-wise polynomial cubic splines at off-knots points
  - solve quantity and prices at outer loop
    * over 10,000 simulations
    * using value functions from inner loop
    * using actual distribution of firms
  - update forecasting rules for m' and p

Steady State.

- Inverse relation b/w firm's capital stocks and their financial savings
  - unconstrained, older, wealthier firms $\to$ minimum saving policy
  - constrained firms have lower capital or lower saving
  - no-constraint[33] firms adopt b/k levels in proportion to k (*assumed*)

[33] We identify no-constraint firms as a type that never faces borrowing constraint.

- Entrants with common $\Phi(\varepsilon)$ but low (b, k)
  - absence of borrowing constraint → jump to k^u with same ε
  - with borrowing constraint → gradual adjustment of k
  - borrow to grow at maximum = binding borrowing constraint
  - long survival = unconstrained firms
- Firm dynamics
  - firm size distribution is right-skewed
  - age ↑ → employment growth ↓
  - larger and older firms pay more dividends
  - "age effects"
- Misallocation:
  - k of young (constrained) firms < k of old (unconstrained) firms
  - should be "=" absent financial frictions
  - old firms do not carry excess capital
  - young, small firms carry too little

Business Cycle.

- Role of credit shocks (7 % of years):
  - reduce aggregate level of y, k and c
  - raise volatility of y, and relative volatility of c, i and n
  - weaken corr(X,y). X= [c, i, n]
  - real shocks dominates
  - more pronounced *conditional on occurrence*
- Evidence: in the crisis,
  - initial ↑ in [c] and ultimate ↓ in [y, n, i] unlike in RBC models
  - *non-contemporaneous* ↓ across [y, n, i, z] unlike in RBC models
  - sharp ↓ in [b] unlike in RBC models
- An 88% drop in collateral value
  - 26% implied reduction in debt
  - expected duration: 3.2 yrs
- Y ↓ immediately by 1.5%
  - capital predetermined
  - labor ↓ by 2.5% ⇐ reduction in expected return to investment ↓
- consumption: ↑ → ↓
  - initial ↑: due to ↓ in return to saving
  - subsequent ↓: due to ↓ in n, y, w (as misallocation ↑)
- unconstrained firms → constrained firms
  - 17% constrained → 43% constrained
  - young firms: slower to catch up with their productivity
- TFP ↓: endogenous !
  - # of medium-size firms ↓; small firms ↑ and very largest firms ↑
    - medium firm: unconstrained → constrained
    - small firm: takes longer to grow
    - largest firm: unconstrained, gain from ↓ r
  - Increased efficiency from small firms
    - widened gap b/w expected investment return and interest rate
    - coexistence of ↑ in MPK and ↓ of ex post r

    * coexistence of $\uparrow$ in MPK of SME and $\downarrow$ of MPK of largest firms
- Reminiscent the finding of Eisfeldt and Rampini (06')
    * dispersion in returns to capital $\uparrow$ in recession;
    * benefit of capital reallocation $\uparrow$ in recession;
    * level of capital reallocation $\downarrow$ in recession
- Disproportionately negative impact on smaller and young firms

10. Uncertainty

In what follows I survey three papers on uncertainty shocks in a RBC model and two New Keynesian one. We introduce to this topic by defining uncertainty shock and establishing some stylized fact. An identified uncertainty shock in the data causes significantly negative comovement in output, consumption, investment, and hours worked. We first show that uncertainty shocks alone in standard RBC models (or in general-equilibrium models with flexible prices) usually cannot reproduce this comovement (Bloom et al. (2018)). By contrast, we subsequently show that sticky price models can (easily) generate comovement through countercyclical markups as Basu and Bundick (2017) and Leduc and Liu (2016), the latter which also highlights a option-value channel with searching-and-matching labor market.
Introduction and Stylized Facts. Generally we assume that a firm, indexed by j, produces output in period t according to the following production function: $y_{j,t} = A_t z_{j,t} k_{j,t}^{\alpha} n_{j,t}^{v}, \alpha + v <$
1
where $k_{t,j}$ and $n_{t,j}$ denote idiosyncratic capital and labor employed by the firm. Each firm's productivity is a product of two separate processes: an aggregate component, A_t, and an idiosyncratic component, $z_{j,t}$.
We assume that the aggregate and idiosyncratic components of business conditions follow autoregressive processes:
$log(A_t) = \rho^A log(A_{t-1}) + \sigma_{t-1}^A \epsilon_t$ (macroeconomic shocks)
$log(z_{j,t}) = \rho^Z log(z_{j,t-1}) + \sigma_{t-1}^Z \epsilon_{j,t}$ (microeconomic shocks)
We allow σ_t^A and σ_t^Z to vary over time, generating periods of low and high macro and micro uncertainty.
These two shocks are driven by different statistics. Volatility in $z_{j,t}$ implies that cross-sectional dispersion-based measures of firm performance (output, sales, stock market returns, etc.) are time-varying, while volatility in At induces higher variability in aggregate variables like GDP growth and the S&P500 index.
In addition to uncertainty shock in production, some literature also models uncertainty by imposing time-varying second moment to the preference shocks.
Regardless of measures or proxies, uncertainty is countercyclical.

10.1. * **Bloom et al. (2018, ECMA).**

General Equilibrium Model with Flexible Price. The model departs from frictionless standard RBC models in three ways:

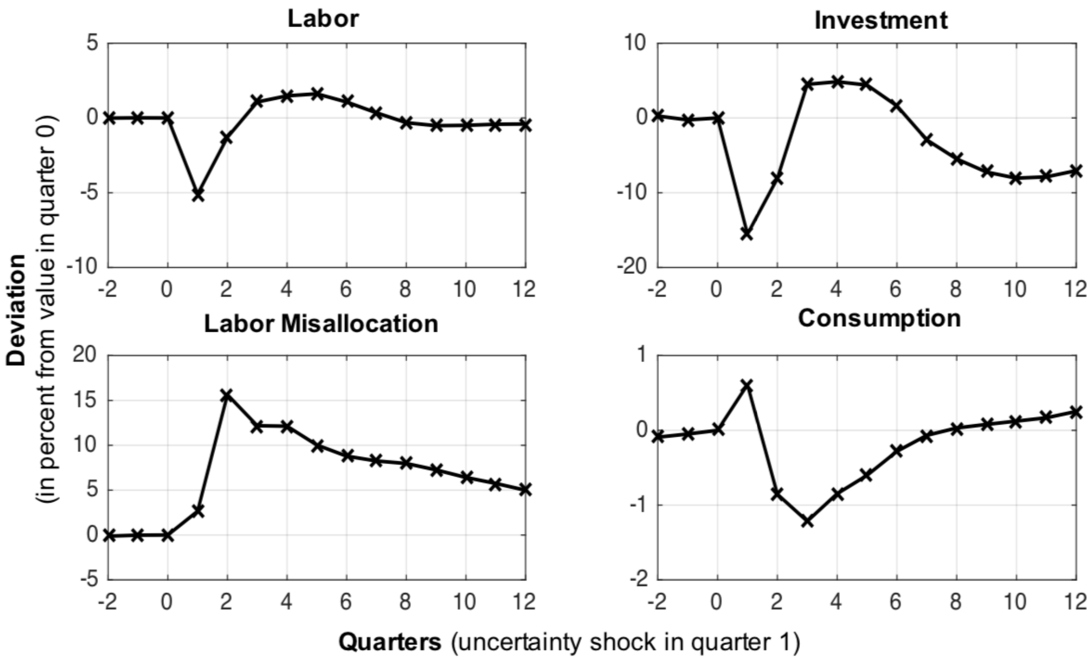
- Uncertainty is time-varying: inclusion of shocks to both the level of technology (first moment) and its variance (second moment), at both microeconomic and macroeconomic levels;
- Heterogeneous firms, subject to idiosyncratic shocks;
- Non-convex adjustment costs in both capital and labor.

We discuss details of the setting.

- **Production technology:** diminishing returns to scale :
  $y_{j,t} = A_t z_{j,t} k_{j,t}^{\alpha} n_{j,t}^{v}, \alpha + v < 1$
  y: firm's output; k&n: idiosyncratic capital & labor;
  Productivity: A_t, aggregate component; $z_{j,t}$, idiosyncratic component.
- AR(1) processes of two components (first moment):
  $log(A_t) = \rho^A log(A_{t-1}) + \sigma_{t-1}^A \epsilon_t$ (macroeconomic shocks)
  $log(z_{j,t}) = \rho^Z log(z_{j,t-1}) + \sigma_{t-1}^Z \epsilon_{j,t}$ (microeconomic shocks)
- We allow σ_t^A and σ_t^Z to vary over time according to a two-state Markov chain.(second moment)
  The two-state Markov chain process of uncertainty[34]:
  $\sigma_t^A \in [\sigma_L^A, \sigma_H^A]$, where $Pr(\sigma_{t+1}^A = \sigma_j^A | \sigma_t^A = \sigma_k^A) = \pi_{k,j}^{\sigma}$
  $\sigma_t^Z \in [\sigma_L^Z, \sigma_H^Z]$, where $Pr(\sigma_{t+1}^Z = \sigma_j^Z | \sigma_t^Z = \sigma_k^Z) = \pi_{k,j}^{\sigma}$
  There are six uncertainty parameters:$\sigma_L^A, \sigma_H^A, \sigma_L^Z, \sigma_H^Z, \pi_{L,H}^{\sigma}, \pi_{H,L}^{\sigma}$
- **Capital Law of Motion:**
  $k_{j,t+1} = (1-\delta_k) k_{j,t} + i_{j,t}$
  where δ_k denotes depreciation rate of capital and i_t denotes net investment.
- subject to capital adjustment cost:
  if $i > 0$, $AC^k = y(z, A, k, n) F^K$;
  if $i < 0$, $AC^k = y(z, A, k, n) F^K + S|i|$;
  where F^K is a fixed disruption cost, $S|i|$ is resale loss for disinvestment (when $i < 0$).
- **Hours Law of Motion:**
  $n_{j,t+1} = (1-\delta_n) n_{j,t} + s_{j,t}$
  where δ_n denotes exogenous destruction rate of hours worked (for example illness, retirement etc.)
  $s_{j,t}$ denotes net flows into hours worked.
- subject to labor adjustment cost:
  if $|s| > 0$,$AC^n = y(z, A, k, n) F^L + |s| Hw$;
  where F^L is a fixed disruption cost, $|s|Hw$ is a linear hiring/firing cost (Hw is aggregate wage).

Effect of Uncertainty Shock. A pure uncertainty shock leads to real effect on macroeconomic aggregates due to existence of non-convex adjustment cost. (Second moment shock alone has limited effect on macroeconomic aggregates if there is no adjustment cost.)

[34]We assume micro- and macro- uncertainty follow the same process.



Three channels of uncertainty shock:

- labor: uncertainty increases, most firms pause recruitment, and because workers continue to leave for illness, maternity or retirement without being replaced, total hours drop.
- investment: investment falls but capital continues to depreciate, there will be a drop in the capital stock
- misallocation: In normal times, unproductive firms contract in size by layoff or by cutting down branches, and productive firms continue to expand through recruitment and setting up new branches, and this mechanism helps maintain high levels of aggregate productivity. But when uncertainty is high, both productive and unproductive firms reduce expansion and contraction, which shuts off the mechanism of reallocation for economic adjustment

Failure of flexible price models in uncertainty shock:

- rise in consumption
- investment $\downarrow$ + output = $\rightarrow$ consumption $\uparrow$

Possible solution for comovement

- open economy approach: allow save in other technologies besides capital, for example, in foreign assets as Fernández-Villaverde et al. (2011)
- preference: complementary preference in consumption and labor
- compound shock: add a first-moment shock as Bloom et al. (2018)
- sticky price: New Keynesian environment with demand-determined output as Basu and Bundick (2017)

10.2. *** Basu and Bundick (2017, ECMA).** This paper solves the comovement issue in uncertainty shock in a New-Keynesian framework with sticky price, thus the economy in the short term is demand-driven.

Flexible price model. Equivalently speaking, a large class of one-sector business-cycle models can be characterized by a few key equations:

$$Y_t = C_t + I_t; \tag{275}$$

$$Y_t = F(K_t, Z_t N_t); \tag{276}$$

$$\frac{W_t}{Pt} U_1(C_t, 1 - N_t) = U_2(C_t, 1 - N_t); \tag{277}$$

$$\frac{W_t}{Pt} = Z_t F_2(K_t, Z_t N_t). \tag{278}$$

Standard neoclassical GE models: no comovement

- uncertainty ↑ → saving ↑ & C ↓ today
- constant N + constant TFP + predetermined K → constant Y today
- I must ↑ today

NK Model. Uncertainty shocks can easily generate comovement by adding countercyclical markups through sticky prices.

Intuition: When prices adjust slowly, by contrast, aggregate demand determines output in the short run, which reverses the causal ordering of these equations. Higher uncertainty reduces the demand for consumption goods, which lowers output directly in Equation (1). Lower output reduces the benefit to owning capital, since the marginal revenue product of capital falls. The decline in the desired capital stock is reflected in a lower level of investment. Since consumption and investment both fall, output and hours worked both decline, since labor is the only input to production that can change in response to higher uncertainty.
In sticky price models, equation (4) can be written as:

$$\frac{W_t}{Pt} = \frac{1}{\mu_t} Z_t F_2(K_t, Z_t N_t). \tag{279}$$

where μ_t is markup in price over marginal cost. Following previous paragraph, mechanically, precautionary labor supply reduces firm marginal cost, which increases the markup when prices are sticky. Thus, equilibrium hours worked may fall as a result of the shifts in labor supply and labor demand, when firm markups increase enough to produce a decrease in equilibrium hours worked in response to a rise in uncertainty.

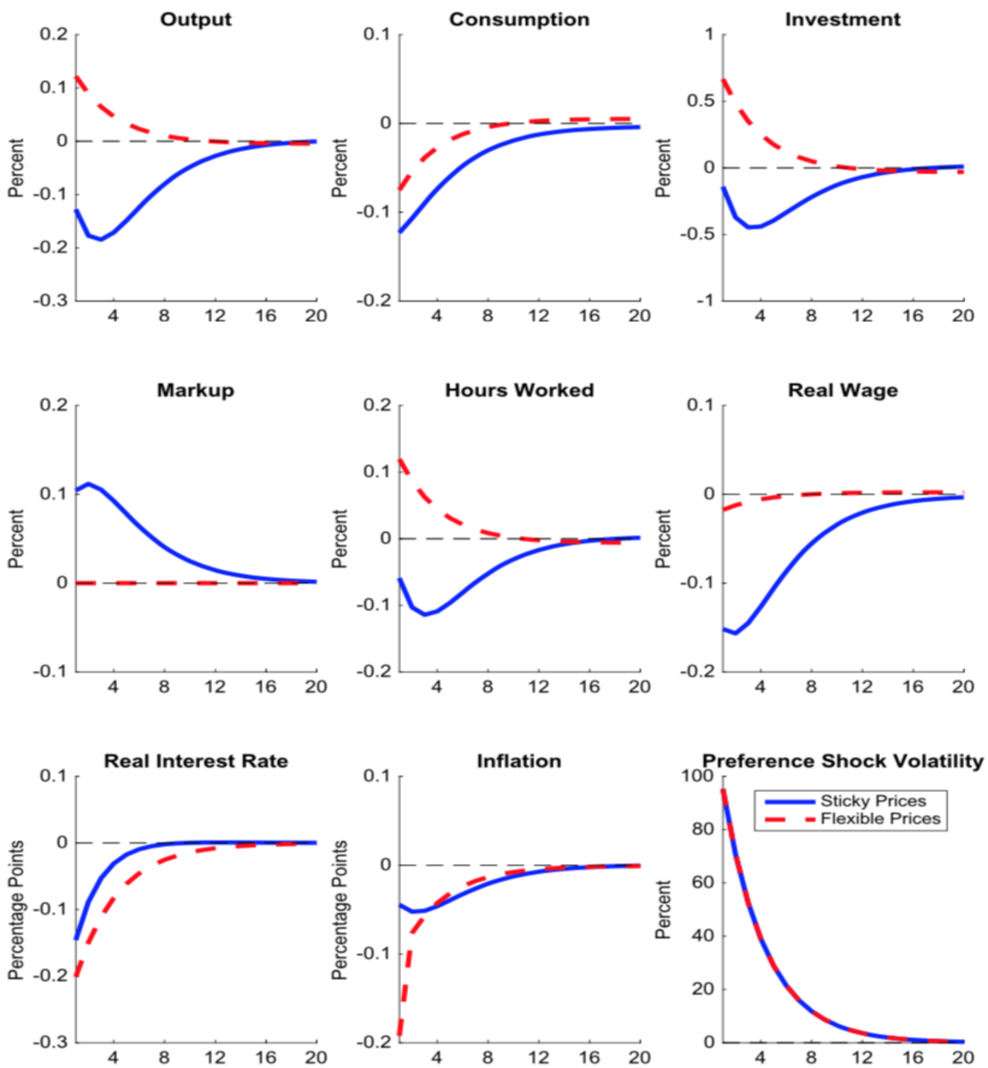


FIGURE 3.—Model-implied impulse responses under flexible and sticky prices.

Effect of Uncertainty Shock: Co-movement. Figure 3 in the paper plots the impulse responses of the model to a demand uncertainty shock under both flexible and sticky prices.

- precautionary motive: households save ↑ by consumption ↓ + hours worked ↑
- demand channel: consumption ↓ → output ↓ → MPK ↓ → capital ↓
- labor demand channel: capital ↓ → labor demand ↓
- low inflation

TABLE II

EMPIRICAL AND MODEL-IMPLIED VOLATILITY IN MACROECONOMIC AGGREGATES[a]

	Percent		Relative to Output	
Moment	Data	Model	Data	Model
Unconditional Volatility				
Output	1.1	1.0	1	1
Consumption	0.7	0.8	0.6	0.7
Investment	3.8	4.7	3.4	4.5
Hours Worked	1.4	0.8	1.3	0.8
Stochastic Volatility				
Output	0.4	0.2	1	1
Consumption	0.2	0.2	0.5	0.7
Investment	1.6	1.2	3.6	5.0
Hours Worked	0.5	0.2	1.0	0.9

[a] Unconditional volatility is measured with the sample standard deviation. We measure stochastic volatility using the standard deviation of the time-series estimate for the 5-year rolling standard deviation. The empirical sample period is 1986–2014. See Appendix A for additional details.

The model closely matches the volatility of output, consumption, and investment we observe in the data. As with many other standard macroeconomic models, however, the model does struggle to generate sufficient fluctuations in hours worked relative to output.

10.3. *** Leduc and Liu (2016, JME).** The paper highlights two channels through which uncertainty shocks affect real economy: an aggregate demand channel due to nominal rigidity, and an option-value channel due to search-and-matching at labor market. We focus on the real option-value channel.

Setting. Household. The representative household consumes a basket of retail goods. The utility function is given by

$$E\sum_{t=0}^{\infty}\beta^t\left[\ln\left(C_t\right)-\chi N_t\right]$$

subject to the sequence of budget constraints

$$C_t+\frac{B_t}{P_tR_t}=\frac{B_{t-1}}{P_t}+w_tN_t+d_t,\quad \forall t\geq 0$$

where P_t denotes the price level, B_t denotes holdings of a nominal risk-free bond, R_t denotes the nominal interest rate, w_t denotes the real wage rate, d_t denotes profit income from ownership of intermediate goods producers and of retailers. Optimal bond-holding decisions are described by the intertemporal Euler equation

$$1=\mathrm{E}_t\beta\frac{\Lambda_{t+1}}{\Lambda_t}\frac{R_t}{\pi_{t+1}}$$

where $\pi_t=\frac{P_t}{P_{t-1}}$ is inflation rate, and Λ_t is marginal utility of consumption.

Aggregation sector. Denote by Y_t the final consumption good, which is a basket of differentiated retail goods. Denote by $Y_t(j)$ a type j retail good for $j\in[0,1]$. We assume that

$$Y_t=\left(\int_0^1 Y_t(j)\frac{\eta-1}{\eta}\right)^{\frac{\eta}{\eta-1}}$$

where $\eta>1$ is the elasticity of substitution between differentiated products. Expenditure minimizing implies that demand for a type j retail good is inversely related to the relative price, with the demand schedule given by

$$Y_t^d(j)=\left(\frac{P_t(j)}{P_t}\right)^{-\eta}Y_t$$

$$P_t=\left(\int_0^1 P_t(j)^{\frac{1}{1-\eta}}\right)^{1-\eta}$$

Retail goods producers. There is a continuum of retailers, each producing a differentiated product using a homogeneous intermediate good as input. The production function of a retail good of type $j\in[0,1]$ is given by

$$Y_t(j)=X_t(j)$$

where $X_t(j)$ is the input of intermediate goods used by retailer j and $Y_t(j)$ is the output. The retail goods producers are price takers in the input market and monopolistic competitors in the product markets, where price adjustments are subject to the quadratic cost

$$\frac{\Omega_p}{2}\left(\frac{P_t(j)}{\pi P_{t-1}(j)}-1\right)^2Y_t,$$

where the parameter $\Omega_p \geq 0$ measures the cost of price adjustments and π denotes the steady-state inflation rate. Price adjustment costs are in units of aggregate output. A retail firm that produces good j chooses $P_t(j)$ to maximize the profit

$$\mathrm{E}_t \sum_{i=0}^{\infty} \frac{\beta^i \Lambda_{t+i}}{\Lambda_t} \left[\left(\frac{P_{t+i}(j)}{P_{t+i}} - q_{t+i} \right) Y_{t+i}^d(j) - \frac{\Omega_p}{2} \left(\frac{P_{t+i}(j)}{\pi P_{t+i-1}(j)} - 1 \right)^2 Y_{t+i} \right]$$

where q_t denotes the relative price of intermediate goods. The optimal price-setting decision implies that, in a symmetric equilibrium with $P_t(j) = P_t$ for all j, we have

$$q_t = \frac{\eta - 1}{\eta} + \frac{\Omega_p}{\eta} \left[\frac{\pi_t}{\pi} \left(\frac{\pi_t}{\pi} - 1 \right) - \mathrm{E}_t \frac{\beta \Lambda_{t+1}}{\Lambda_t} \frac{Y_{t+1}}{Y_t} \frac{\pi_{t+1}}{\pi} \left(\frac{\pi_{t+1}}{\pi} - 1 \right) \right].$$

Labor Market. In the beginning of period t, there are u_t unemployed workers searching for jobs and there are v_t vacancies posted by firms. The matching technology is described by the Cobb-Douglas function

$$m_t = \mu u_t^{\alpha} v_t^{1-\alpha}$$

where m_t denotes the number of successful matches and the parameter $\alpha \in (0,1)$ denotes the elasticity of job matches with respect to the number of searching workers. The parameter μ scales the matching efficiency. The probability that an open vacancy is matched with a searching worker (i.e., the job filling rate) is given by

$$q_t^v = \frac{m_t}{v_t}$$

The probability that an unemployed and searching worker is matched with an open vacancy (i.e., the job finding rate) is given by

$$q_t^u = \frac{m_t}{u_t}$$

In the beginning of period t, there are N_{t-1} workers. A fraction ρ of these workers lose their jobs. Thus, the number of workers who survive the job separation is $(1-\rho)N_{t-1}$. At the same time, m_t new matches are formed. Thus, aggregate employment in period t evolves according to

$$N_t = (1-\rho)N_{t-1} + m_t.$$

With a fraction ρ of employed workers separated from their jobs, the number of unemployed workers searching for jobs in period t is given by

$$u_t = 1 - (1-\rho)N_{t-1}$$

The unemployment rate is given by

$$U_t = u_t - m_t = 1 - N_t$$

Labor Demand A firm can produce only if it successfully hires a worker. The production function for a firm with one worker is given by

$$x_t = Z_t,$$

where x_t denotes output. The term Z_t denotes an aggregate technology shock, which follows the stationary stochastic process

$$\ln Z_t = \rho_z \ln Z_{t-1} + \sigma_{zt} \varepsilon_{zt}$$

The parameter $\rho_z \in (-1,1)$ measures the persistence of the technology shock. The term ϵ_{zt} is an i.i.d. innovation to the technology shock and is a standard normal process. The term σ_{zt} is a time-varying standard deviation of the innovation, which we interpret as a technology uncertainty shock. We assume that the uncertainty shock follows the stationary stochastic process

$$\ln \sigma_{zt} = (1-\rho_{\sigma_z}) \ln \sigma_z + \rho_{\sigma_z} \ln \sigma_{z,t-1} + \sigma_{\sigma_z} \varepsilon_{\sigma_z,t},$$

where the parameter $\rho_{\sigma_z} \in (-1,1)$ measures the persistence of the uncertainty shock, the term $\varepsilon_{\sigma_2,t}$ is an i.i.d. standard normal process, and the parameter $\sigma_{\sigma_z} > 0$ is the standard deviation of the innovation to technology uncertainty.

If a firm finds a match, it obtains a flow profit in the current period after paying the worker. In the next period, if the match survives (with probability $1-\rho$), the firm continues; if the match breaks down (with probability ρ), the firm posts a new job vacancy at a fixed cost κ, with the value V_{t+1}. The value of a firm with a match (denoted by J_t^F) is therefore given by the Bellman equation

$$J_t^F = q_t Z_t - w_t + E_t \frac{\beta \Lambda_{t+1}}{\Lambda_t} \left[(1-\rho) J_{t+1}^F + \rho V_{t+1} \right].$$

If the firm posts a new vacancy in period t, it costs κ units of final goods. The vacancy can be filled with probability q_t^v, in which case the firm obtains the value of the match. Otherwise, the vacancy remains unfilled and the firm goes into the next period with the value V_{t+1}. Thus, the value of an open vacancy is given by

$$V_t = -\kappa + q_t^v J_t^F + E_t \frac{\beta \Lambda_{t+1}}{\Lambda_t} \left(1 - q_t^v\right) V_{t+1}$$

Free entry implies that $V_t = 0$, so that

$$\frac{\kappa}{q_t^v} = J_t^F$$

Labor Supply. If a worker is employed, he obtains wage income but suffers a utility cost of working. In period $t+1$, the match is separated with probability ρ and the separated worker can find a new match with probability q_{t+1}^u. Thus, with probability $\rho\left(1-q_{t+1}^u\right)$, a separated worker fails to find a new job in period $t+1$ and enters the unemployment pool. Otherwise, the worker continues to be employed. The (marginal) value of an employed worker (denoted by J_t^W) therefore satisfies the Bellman equation

$$J_t^W = w_t - \frac{\chi}{\Lambda_t} + E_t \frac{\beta \Lambda_{t+1}}{\Lambda_t} \left\{ \left[1 - \rho\left(1-q_{t+1}^u\right)\right] J_{t+1}^W + \rho\left(1-q_{t+1}^u\right) J_{t+1}^U \right\}$$

where J_t^U denotes the value of an unemployed worker. An unemployed worker obtains nothing and can find a new job in period $t+1$ with probability q_{t+1}^u. Thus, the value of an unemployed worker satisfies the Bellman equation

$$J_t^U = E_t \frac{\beta \Lambda_{t+1}}{\Lambda_t} \left[q_{t+1}^u J_{t+1}^W + \left(1-q_{t+1}^u\right) J_{t+1}^U \right]$$

Wage Determination. Firms and workers bargain over wages. The Nash bargaining problem is given by

$$\max_{w_t} \left(J_t^W - J_t^U\right)^b \left(J_t^F\right)^{1-b}$$

where $b \in (0,1)$ represents the bargaining weight for workers. Define the total surplus as

$$S_t = J_t^F + J_t^W - J_t^U$$

Then the bargaining solution is given by

$$J_t^F = (1-b) S_t, \quad J_t^W - J_t^U = b S_t$$

It then follows that

$$bS_t = w_t^N - \frac{\chi}{\Lambda_t} + E_t \frac{\beta\Lambda_{t+1}}{\Lambda_t} \left[(1-\rho)\left(1-q_{t+1}^u\right) bS_{t+1}\right]$$

Given the bargaining surplus S_t, which itself is proportional to the match value J_t^F, this last equation determines the Nash bargaining wage w_t^N.

If the equilibrium real wage equals the Nash bargaining wage, then we can obtain an explicit expression for the Nash bargaining wage.

$$w_t^N = (1-b)\left[\frac{\chi}{\Lambda_t} + \phi\right] + b\left[q_t Z_t + \beta(1-\rho) E_t \frac{\beta\Lambda_{t+1}}{\Lambda_t} \frac{\kappa v_{t+1}}{u_{t+1}}\right].$$

We follow the literature to formalize sticky wage by assuming that

$$w_t = w_{t-1}^{\gamma} \left(w_t^N\right)^{1-\gamma}$$

where $\gamma \in (0,1)$ represents the degree of real wage rigidity.

Policy. The monetary authority follows the Taylor rule

$$R_t = r\pi^* \left(\frac{\pi_t}{\pi^*}\right)^{\phi_\pi} \left(\frac{Y_t}{Y}\right)^{\phi_y}$$

where the parameter ϕ_π determines the aggressiveness of monetary policy against deviations of inflation from the target π^* and ϕ_y determines the extent to which monetary policy accommodates output fluctuations. The parameter r denotes the steady-state real interest rate (i.e., $r = \frac{R}{\pi}$).

In a search equilibrium, the markets for bonds, final consumption goods, and intermediate goods all clear.

Real Option-Value channel of Uncertainty Shock. $\Omega_p = 0$: flexible-price model

- spot labor market (w/o searching):
  - uncertainty ↑ → real interest rate ↓ → PV of job match ↑ → N ↑ → Y ↑
- searching labor market:
  - job match: irreversible long-term employment relation
  - uncertainty ↑ → option-value of waiting ↑ → EV of job match ↓ → job posting ↓ → unemployment ↑ → Y ↑

10.4. *** Arellano, Bai, and Kehoe (2019, JPE).** This paper proposes a theory of uncertainty shock that connects rise in volatility at firm-level to economic and financial recession. The model features irreversible labor hiring decision and incomplete financial market.

Settings. We presents a two-period stylized model to illustrate the intuition.

- date 0:
  - technology:
$$y = l^{\alpha}$$
  - demand: (z unknown)
$$y = \left(\frac{z}{p}\right)^{\eta} Y$$
  - l: irreversible labor hiring (chosen before knowing z)
  - b: outstanding debt
  - z follows Markov process with volatility σ
- date 1:
  - z: realized
  - p: chosen to maximize profit
  - V: continuation value assumed to be constant
  - d: equity (dividend)
$$d = py - wl - b \geq 0$$

Scenario I: complete financial market. The firm chooses labor and state-contingent debt to solve the following problem:

$$\max_{\ell, b(z)} \int_0^{\infty} \left[p(z,\ell)\ell^{\alpha} - w\ell - b(z)\right] \pi_z(z) dz + V$$

s.t. IR constraint of creditors

$$\int_0^{\infty} b(z)\pi_z(z)dz = b$$

and s.t. non-negative equity payout condition

$$p(z,\ell)\ell^{\alpha} - w\ell - b(z) \geq 0,$$

where $p(z,\ell) = zY^{1/\eta}\ell^{-\alpha/\eta}$. [35]

With complete markets, the firm's optimal labor choice ℓ^* is such that the expected marginal product of labor is a constant markup over the wage:

$$E\left(p\left(z,\ell^*\right)\right)\alpha\left(\ell^*\right)^{\alpha-1} = \frac{\eta}{\eta-1}w$$

which implies that volatility in z has no effect on labor choice.

[35] Assume that the initial debt b is small enough so that with complete financial markets, the firm can guarantee positive cash flows in every state by using state-contingent debt $b(z)$, and the equity payout constraint is not binding.

Scenario I: incomplete financial market. Now assume that financial market is incomplete such that debts are state un-contingent. This implies that, upon realization of bad idiosyncratic shock, firm defaults and exits the market. The cut-off is given as

$$p(\hat{z}, l)l^{\alpha} - wl - b = 0$$

Now firms maximize

$$\max_{\ell,\hat{z}} \int_{\hat{z}}^{\infty} \left[p(z,\ell)z\ell^{\alpha} - w\ell - b\right]\pi_z(z)dz + \int_{\hat{z}}^{\infty} V\pi_z(z)dz$$

The optimal labor choice satisfies

$$E\left(p\left(z,\ell^*\right) \mid z \geq \hat{z}\right)\alpha\left(\ell^*\right)^{\alpha-1} = \frac{\eta}{\eta-1}\left(w + V\frac{\pi_z(\hat{z})}{1-\Pi_z(\hat{z})}\frac{d\hat{z}}{d\ell^*}\right)$$

where $p\left(\hat{z},\ell^*\right)\left(\ell^*\right)^{\alpha} - w\ell^* - b = 0$ and $\Pi_z(z)$ is the distribution function associated with the density $\pi_z(z)$.

This implies that when financial market is incomplete, volatility in z has real effect on labor choice, such that firms equates the effective marginal product of labor conditional on not default, to the marginal costs arising from increasing labor, which includes the wage and the loss in future value upon default.

10.5. *** Dong, Liu and Wang (2021, WP).** In this task we solve a simplified model of Dong, Liu and Wang (2021) with flexible price. The model features heterogeneous firms and highlights a misallocation of uncertainty shock to generate comovement.
Setting. Consider an economy with a continuum of firms that produce with a linear technology using labor n_{jt} as single input:

$$y_{jt} = A_t z_{jt} n_{jt} \tag{280}$$

where A_t measures aggregate productivity, and z_{jt} measures idiosyncratic productivity.

The process of idiosyncratic productivity is assumed to follow the following process:

$$z_{jt+1} = \begin{cases} z_{jt} & w.p. \quad \rho_t \\ \tilde{z} & w.p. \quad 1-\rho_t \end{cases} \tag{281}$$

where $\tilde{z}$ is discrete random variable with $\tilde{z} = z_j$ occurring with probability π_j, $j = 1, 2, ..., I$. We assume that $z_1 < z_2 < ... < z_I$ without loss of generality. The process features time-invariant cross-sectional distribution of firm productivity such that, regardless of realization on ρ_t, there are always π_j fraction of firms with $z_{jt} = z_j$ in each period.

The firms' problem is given by the following Bellman equation (we suppress aggregate state in notation for simple exposition):

$$V_t(z_{jt}, \tau_{jt}) = \tau_{jt} A_t z_{jt} n_{jt} - W_t n_{jt} + \beta E_t V_{t+1}(z_{jt+1}, \tau_{jt+1}) \tag{282}$$

subject to a credit constraint

$$W_t n_{jt} \leq \theta_t \beta E_t V_{t+1}(z_{jt+1}, \tau_{jt+1}) \tag{283}$$

where θ_t is a financial shock measuring tightness of credit constraint. τ_{jt} is idiosyncratic distortion (net subsidy) on output, and is assumed to be an i.i.d. random variable with cumulative distribution function $F(\tau)$. Denote

$$\bar{V}_{jt} = \int V_t(z_{jt}, \tau_{jt}) dF(\tau), \tag{284}$$

we can write discounted future value conditional on current realization of productivity as

$$\beta E_t V_{t+1}(z_{jt+1}, \tau_{jt+1} | z_{jt} = z_j) = \beta E_t \left[\rho_t \bar{V}_{jt+1} + (1-\rho_t) \sum_{i=1}^{I} \pi_i \bar{V}_{it+1} \right] \equiv B_{jt} \tag{285}$$

Solving static profit maximization problem gives the allocations of production and credit: a firm will borrow and produce in current period if and only if net subsidy τ_{jt} is higher than a cut-off:

$$\hat{\tau}_{jt} \equiv \hat{\tau}_t(z_{jt}) = \frac{W_t}{A_t z_{jt}} \tag{286}$$

Without loss of generality we assume marginal firms operate. Firms with relatively higher productivity (z_{jt}) and subsidy (or lower tax) choose to produce and borrow up to the limit to finance wage bill. Low productivity or heavily taxed firms stay inactive and do not borrow. Therefore, idiosyncratic labor demand function is

$$n_t(z_{jt}, \tau_{jt}) = \begin{cases} \frac{\theta_t B_{jt}}{W_t}, & if \quad \tau_{jt} \geq \hat{\tau}_{jt} \\ 0, & otherwise \end{cases} \tag{287}$$

The value function in equation (282) can be re-written as

$$V_t(z_{jt}, \tau_{jt}) = \max(\frac{A_t z_{jt} \tau_{jt}}{W_t} - 1, 0) \theta_t B_{jt} + B_{jt} \tag{288}$$

Define aggregate firm value as

$$\bar{V}_t = \sum_{i=1}^{I} \pi_i \bar{V}_{it} \tag{289}$$

It follows that for $j = 1, 2, ..., I$,

$$\bar{V}_{jt} = \left[1 + \theta_t \int_{\frac{W_t}{A_t z_{jt}}} \left(\frac{A_t z_{jt}}{W_t}\tau - 1\right) dF(\tau)\right] \beta E_t \left[\rho_t \bar{V}_{jt+1} + (1-\rho_t)\bar{V}_t\right] \equiv \Phi(\frac{W_t}{A_t z_{jt}}, \theta_t) B_{jt} \tag{290}$$

where $\Phi(\frac{W_t}{A_t z_{jt}}, \theta_t) \equiv 1 + \theta_t \int_{\frac{W_t}{A_t z_{jt}}} \left(\frac{A_t z_{jt}}{W_t}\tau - 1\right) dF(\tau)$. It is clear that Φ is an increasing function of z_{jt}, θ_t, and decreasing function of W_t.
To solve the equilibrium, we need to impose a labor market clearing condition:

$$N_t = \sum_{j=1}^{I} \pi_j \int_{\tau} n_t(z_{jt}, \tau) dF(\tau) \equiv \sum_{j=1}^{I} \pi_j N_{jt}, \tag{291}$$

where N_t is exogenous labor supply and N_{jt} for $j = 1, 2, ..., I$ is

$$N_{jt} = \frac{\theta_t \beta E_t \left[\rho_t \bar{V}_{jt+1} + (1-\rho_t)\bar{V}_t\right]}{W_t} \left[1 - F(\frac{W_t}{A_t z_{jt}})\right] \tag{292}$$

The aggregate output (Y_t) is given by

$$Y_t = \sum_{j=1}^{I} \pi_j \frac{A_t z_{jt} \theta_t B_{jt}}{W_t} [1 - F(\frac{W_t}{A_t z_{jt}})] = A_t \sum_{j=1}^{I} \pi_j z_{jt} N_{jt} \tag{293}$$

and endogenous TFP, denoted as Z_t, is defined as

$$Z_t \equiv \frac{Y_t}{A_t N_t} = \frac{\sum_{j=1}^{I} \pi_j z_{jt} N_{jt}}{N_t} \tag{294}$$

Last equation shows that, given exogenous labor supply, endogenous TFP reflects labor misallocation.

Characterization of stationary equilibrium. We now characterize the stationary equilibrium. Equation (290) at steady state implies that idiosyncratic firm value is a constant share, denoted as g_j, of aggregate firm value:

$$\bar{V}_j = \frac{\beta(1-\rho)\Phi(\frac{W}{A z_j}, \theta)}{1 - \beta\rho\Phi(\frac{W}{A z_j}, \theta)} \bar{V} \equiv g_j \bar{V} \tag{295}$$

Then by definition,

$$\sum_{j=1}^{I} \pi_j \bar{V}_j \equiv \sum_{j=1}^{I} \pi_j g_j \bar{V} = \bar{V}$$

which implies that

$$\sum_{j=1}^{I} \pi_j g_j \equiv \sum_{j=1}^{I} \pi_j \frac{\beta(1-\rho)\Phi(\frac{W}{A z_j}, \theta)}{1 - \beta\rho\Phi(\frac{W}{A z_j}, \theta)} = 1 \tag{296}$$

Last equation solves the steady state wage W.

The following lemma establishes the relation between the endogenous component of TFP (the misallocation effect) and the average value of firm-level uncertainty ρ.

Lemma 10.1. *(Misallocation Effect of Firm-Level Uncertainty) Relative labor share of productive firm to less productive firm increases in ρ.*

Proof. Recall from equation (292) that labor demand from firm j is

$$N_j = \frac{\beta\theta\left[\rho g_j \bar{V} + (1-\rho)\bar{V}\right]}{W}\left[1 - F(\frac{W}{Az_j})\right]$$

We can define relative labor share, denoted as $\eta_{ji} \equiv \frac{N_j/N}{N_i/N}$, as

$$\begin{aligned}\eta_{ji} &= \frac{\beta\theta\left[\rho g_j + (1-\rho)\right]\left[1 - F(\frac{W}{Az_j})\right]}{\beta\theta\left[\rho g_i + (1-\rho)\right]\left[1 - F(\frac{W}{Az_i})\right]} = \frac{\left[\rho\left(\frac{\beta\Phi(\frac{W}{Az_j},\theta)-1}{1-\beta\rho\Phi(\frac{W}{Az_j},\theta)}+1\right)+(1-\rho)\right]\left[1 - F(\frac{W}{Az_j})\right]}{\left[\rho\left(\frac{\beta\Phi(\frac{W}{Az_i},\theta)-1}{1-\beta\rho\Phi(\frac{W}{Az_i},\theta)}+1\right)+(1-\rho)\right]\left[1 - F(\frac{W}{Az_i})\right]} \\ &= \frac{\left[\frac{\rho\beta\Phi(\frac{W}{Az_j},\theta)-\rho}{1-\beta\rho\Phi(\frac{W}{Az_j},\theta)}+1\right]\left[1 - F(\frac{W}{Az_j})\right]}{\left[\frac{\rho\beta\Phi(\frac{W}{Az_i},\theta)-\rho}{1-\beta\rho\Phi(\frac{W}{Az_i},\theta)}+1\right]\left[1 - F(\frac{W}{Az_i})\right]} = \frac{1-\beta\rho\Phi(\frac{W}{Az_i},\theta)}{1-\beta\rho\Phi(\frac{W}{Az_j},\theta)}\frac{\left[1 - F(\frac{W}{Az_j})\right]}{\left[1 - F(\frac{W}{Az_i})\right]}\end{aligned} \tag{297}$$

For $z_j > z_i$, as $\Phi(\frac{W}{Az_j},\theta) > \Phi(\frac{W}{Az_i},\theta)$ and $1 - F(\frac{W}{Az_j}) > 1 - F(\frac{W}{Az_i}) > 0$, we have

$$\frac{\partial\eta_{ji}}{\partial\rho} = \frac{\beta[\Phi(\frac{W}{z_j},\theta) - \Phi(\frac{W}{z_i},\theta)]}{(1-\beta\rho\Phi(\frac{W}{z_j},\theta))^2}\frac{\left[1 - F(\frac{W}{z_j})\right]}{\left[1 - F(\frac{W}{z_i})\right]} > 0 \tag{298}$$

Thus, labor share of productive firm to less productive firm increases in ρ. □

This lemma states that, when ρ decreases (uncertainty rises), labor resources are reallocated to less productive firms. Intuitively, when uncertainty rises (ρ declines) current productivity is less predictive for future productivity, thus productive firms are less likely to stay productive. Thus, higher uncertainty lowers expected value of productive firms, reducing credit limit available to finance labor cost. As a consequence, labor demand of productive firm declines and labor are reallocated to less productive firm.

10.6. * **Bernstein, Plante, Richter and Throckmorton (2021, WP).** This paper proposes a simple explanation for the countercyclical fluctuations in real uncertainty: labor market search and matching frictions.

Aggregate Uncertainty. Business cycle dynamics are driven by shocks to technology (TFP):

$$\ln a_t = (1-\rho_a)\ln \bar{a} + \rho_a \ln a_{t-1} + \sigma_{a,t}\varepsilon_{a,t}, \quad -1<\rho_a<1, \varepsilon_{a,t}\sim \mathbb{N}(0,1). \tag{299}$$

The second driving force determines the volatility of TFP shocks, which follows an independent process

$$\ln \sigma_{a,t} = (1-\rho_{sv})\ln \bar{\sigma}_a + \rho_{sv}\ln \sigma_{a,t-1} + \sigma_{sv}\varepsilon_{sv,t}, -1<\rho_{sv}<1, \varepsilon_{sv,t}\sim \mathbb{N}(0,1). \tag{300}$$

Therefore, TFP is subject to volatility shocks, $\varepsilon_{sv,t}$, that exogenously determine the time-variation standard deviation of TFP shock.

Searching and Matching. Entering period t, there are n_{t-1} employed workers and $u_{t-1} = 1-n_{t-1}$ unemployed workers. A fraction $\bar{s}$ of employed workers then exogenously lose their jobs. A fraction $\chi \in [0,1]$ of newly separated workers start searching for jobs in period t. Therefore, the mass of unemployed searching workers in period t is

$$u_t^s = u_{t-1} + \chi \bar{s} n_{t-1}. \tag{301}$$

If the firm posts v_t vacancies, the matching process is described by the Cobb-Douglas function,

$$\mathcal{M}(u_t^s, v_t) = \xi (u_t^s)^{\phi} v_t^{1-\phi} \tag{302}$$

$$m_t = \min\{\mathcal{M}(u_t^s, v_t), u_t^s, v_t\} \tag{303}$$

where $\xi>0$ is matching efficiency and $\phi \in (0,1)$ is the elasticity of matches with respect to unemployed searching. The employment law of motion, job finding rate, and job filling rate are given by

$$n_t = (1-\bar{s})n_{t-1} + m_t \tag{304}$$

$$f_t = m_t/u_t^s \tag{305}$$

$$q_t = m_t/v_t \tag{306}$$

Households. The representative households are fully insured against idiosyncratic risk. They choose consumption, investment, and capital to solve

$$J_t^H = \max_{c_t,i_t,k_t} \ln c_t + \beta E_t\left[J_{t+1}^H\right] \tag{307}$$

subject to

$$k_t = (1-\delta)k_{t-1} + \left(a_1 + \frac{a_2}{1-1/\nu}\left(\frac{i_t}{k_{t-1}}\right)^{1-1/\nu}\right)k_{t-1}, \tag{308}$$

where $0<\delta\leq 1$ is the capital depreciation rate, $\nu>0$ determines the size of the capital adjustment cost, and $a_1 = \delta/(1-\nu)$ and $a_2 = \delta^{1/\nu}$ are chosen so there are no adjustment costs in steady state, and

$$c_t + i_t = w_t n_t + r_t^k k_{t-1} + b u_t - \tau_t \tag{309}$$

where w_t is the wage rate, r_t^k is the rental rate, b is the flow value of unemployment, and τ_t is a lump-sum tax. Letting $x_{t+1} = \beta\left(c_t/c_{t+1}\right)$ denote the household's pricing kernel, F.O.C. is

$$\frac{1}{a_2}\left(\frac{i_t}{k_{t-1}}\right)^{1/\nu} = E_t\left[x_{t+1}\left(r_{t+1}^k + \frac{1}{a_2}\left(\frac{i_{t+1}}{k_t}\right)^{1/\nu}(1-\delta+a_1) + \frac{1}{\nu-1}\frac{i_{t+1}}{k_t}\right)\right]. \tag{310}$$

Firms. The representative firm combines capital and labor to produce the final good with a Cobb-Douglas production function, $y_t = a_t k_{t-1}^{\alpha} n_t^{1-\alpha}$. It posts vacancies at cost κ to attract new workers. The firm chooses capital, employment, and vacancies to solve

$$J_t^F = \max_{k_{t-1}, n_t, v_t} a_t k_{t-1}^{\alpha} n_t^{1-\alpha} - w_t n_t - r_t^k k_{t-1} - \kappa v_t + E_t\left[x_{t+1} J_{t+1}^F\right] \tag{311}$$

subject to

$$n_t = (1-\bar{s})n_{t-1} + q_t v_t \tag{312}$$

$$v_t \geq 0 \tag{313}$$

Letting $\lambda_{n,t}$ denote the Lagrange multiplier on the law of motion for employment and $\lambda_{v,t}$ denote the multiplier on the non-negativity constraint for vacancies, the optimality conditions are given by

$$r_t^k = \alpha y_t / k_{t-1} \tag{314}$$

$$\lambda_{n,t} = (1-\alpha)y_t/n_t - w_t + (1-\bar{s})E_t\left[x_{t+1}\lambda_{n,t+1}\right] \tag{315}$$

$$q_t \lambda_{n,t} = \kappa - \lambda_{v,t} \tag{316}$$

$$\lambda_{v,t} v_t = 0, \quad \lambda_{v,t} \geq 0. \tag{317}$$

Wages. Wages are determined via Nash bargaining between employed workers and the firm. Let $\eta \in [0,1]$ denote a worker's bargaining weight and define $\theta_t = v_t/u_t^s$ as labor market tightness.

$$w_t = \eta\left((1-\alpha)y_t/n_t + \kappa(1-\chi\bar{s})E_t\left[x_{t+1}\theta_{t+1}\right]\right) + (1-\eta)b. \tag{318}$$

Equilibrium. Setting $\tau_t = b u_t$, the aggregate resource constraint is given by

$$c_t + i_t + \kappa v_t = a_t k_{t-1}^{\alpha} n_t^{1-\alpha}. \tag{319}$$

The equilibrium consists of infinite sequences of quantities $\{k_t, c_t, n_t, i_t, u_t^s, v_t, m_t, \mathcal{M}_t, q_t, f_t\}_{t=0}^{\infty}$, prices $\left\{w_t, r_t^k, \lambda_{n,t}, \lambda_{v,t}\right\}_{t=0}^{\infty}$, and exogenous variables $\{a_t, \sigma_{a,t}\}_{t=0}^{\infty}$ that satisfy aforementioned equations, given an initial state of the economy $\{k_{-1}, n_{-1}, a_{-1}, \sigma_{a,-1}\}$ and the sequences of TFP shocks $\{\varepsilon_{a,t}, \varepsilon_{sv,t}\}_{t=1}^{\infty}$.

Measuring uncertainty. To measure uncertainty, we follow Jurado et al. (2015) and Ludvigson et al. (2021), who define the uncertainty of outcome $y_{j,t}$ as the period- t conditional volatility of its h-period ahead forecast error. We define aggregate uncertainty in the model as the uncertainty of output growth at a quarterly horizon,

$$\mathcal{U}_t = \frac{1}{SD(\Delta y)}\sqrt{E_t\left[\left(\ln y_{t+3} - E_t\left[\ln y_{t+3}\right]\right)^2\right]} \tag{320}$$

We normalize by the standard deviation of output growth in the ergodic distribution, $SD(\Delta y)$, so the units are consistent with our empirical uncertainty measure based on standardized time series.

Endogenous Uncertainty and Search and Matching Frictions.

- labor law of motion:
$$n_{t+1} = (1-\bar{s})n_t + m_{t+1} \tag{321}$$
rearrange and divide both sides by n_t
$$\frac{n_{t+1}-n_t}{n_t} = \frac{m_{t+1}-\bar{s}n_t}{n_t} \tag{322}$$
and it implies
$$Var(\Delta \log n_{t+1}) \approx (\frac{1}{n_t})^2 Var(m_{t+1}) \tag{323}$$
- first channel: matching volatility channel
$$m_{t+1} = q_{t+1}v_{t+1} \tag{324}$$
Since $q_{t+1} = \xi(v_{t+1}/u^s_{t+1})^{-\phi}$, and u^s_{t+1} is predetermined,
$$Var(\Delta \log n_{t+1}) \approx (\frac{1}{n_t})^2 \underbrace{[\xi(u^s_{t+1})^{\phi}]^2}_{\text{Matching Volatility Channel}} Var\left[v_{t+1}^{1-\phi}\right] \tag{325}$$
Employment growth uncertainty is increasing in countercyclical unemployment, given the conditional volatility of vacancies. Intuition: given number of vacancy, $u^s_{t+1}\uparrow \rightarrow$ job filling rate $q_{t+1}\uparrow \rightarrow$ employment more sensitive ($LHS\uparrow$)
- second channel: vacancy volatility channel
$$v_{t+1} = \frac{m_{t+1}}{q_{t+1}} = \frac{\xi\,(u^s_t)^{\phi}\,v_t^{1-\phi}}{q_{t+1}} \tag{326}$$
and $q_{t+1} = \kappa/\lambda_{n,t+1}$ imply $v_{t+1} = u^s_{t+1}(\xi\lambda_{n,t+1}/\kappa)^{1/\phi}$, thus
$$Var(\Delta \log n_{t+1}) \approx (\frac{1}{n_t})^2[\xi(u^s_{t+1})^{\phi}]^2 \underbrace{[(\xi/\kappa)^{(1-\phi)/\phi}\left(u^s_{t+1}\right)^{1-\phi}]^2}_{\text{Vacancy Volatility Channel}} Var\left[\lambda_{n,t+1}^{(1-\phi)/\phi}\right] \tag{327}$$
Employment growth uncertainty is also increasing in unemployment due to vacancy creation. Intuition: $u^s_{t+1}\uparrow \rightarrow$ sensitivity of labor market tightness $\theta_{t+1} \equiv \frac{v_{t+1}}{u^s_{t+1}}$ to vacancy $\downarrow \rightarrow$ responsiveness of the job filling rate and the marginal cost of hiring ($\kappa/q_{t+1} \equiv (\kappa/\xi)\theta^{\phi}$) to vacancy creation $\downarrow$. Therefore, shocks to the marginal benefit of hiring ($\lambda_{n,t+1}$) require larger vacancy creation responses to equate the marginal cost of hiring with the marginal benefit in equilibrium.
- third channel: matching value channel
$$\lambda_{n,t+1} \sim (1-\alpha)a_{t+1}(k_{t+1}/n_{t+1})^{\alpha} \tag{328}$$
Employment growth uncertainty depends on the uncertainty surrounding the value of a new match ($\lambda_{n,t+1}$), which mainly depends on marginal product of labor. MPL response is larger when response of employment n_{t+1} is lower, or when current unemployment is lower. Thus uncertainty about the match value is procyclical.

10.7. *** Atkinson, Plante, Richter,& Throckmorton (2021, RED).** This paper examines the quantitative significant of complementarity between capital and labor in generating time-varying endogenous uncertainty. An estimated real business cycle model with a constant elasticity of substitution (CES) production function and real frictions in the form of habit persistence in consumption and investment adjustment costs can successfully match either the uncertainty moments or the labor share moments, but not both. Adding exogenous volatility shock can match the volatility of uncertainty and labor share dynamics, but a forecast error variance decomposition reveals that endogenous uncertainty explains at most 16% of the variation in aggregate uncertainty.

10.7.1. *Mechanism.* Complementarity between capital and labor inputs in production can generate time-varying endogenous uncertainty because the concavity in the production function influences how output responds to productivity shocks in different states of the economy. For example, a positive labor productivity shock generates a larger change in output when the capital-to-labor ratio is high compared to when the capital-labor ratio is low.

Cobb-Douglas production function. Consider a production technology with two inputs (denoted as K_t and N_t):

$$Y_t = A_t K_t^{\alpha} N_t^{1-\alpha} \tag{329}$$

or in log-form (i.e. $y_t = \log(Y_t)$):

$$y_t = a_t + \alpha k_t + (1-\alpha) n_t \tag{330}$$

Assume TFP and two inputs are random variables involving following:

$$\log(A_t) \equiv a_t = (1-\rho^a) a^{ss} + \rho^a a_{t-1} + \sigma_a \varepsilon_t^a \tag{331}$$

$$\log(K_t) \equiv k_t = (1-\rho^k) k^{ss} + \rho^k k_{t-1} + \sigma_k \varepsilon_t^k \tag{332}$$

$$\log(N_t) \equiv n_t = (1-\rho^n) n^{ss} + \rho^n n_{t-1} + \sigma_n \varepsilon_t^n \tag{333}$$

We can measure uncertainty of log output (growth) one period ahead as:

$$\begin{aligned} U_{t,t+1}^{y} &= E_t\left\{[(y_{t+1}-y_t) - E_t(y_{t+1}-y_t)]^2\right\} = E_t\left\{[y_{t+1} - E_t(y_{t+1})]^2\right\} \\ &= \sigma_a^2 + \alpha^2 \sigma_k^2 + (1-\alpha)^2 \sigma_n^2 \end{aligned} \tag{334}$$

Thus, the conditional volatility of log output (growth) is a weighted average of the variance of each shock. There are four key assumptions behind the results: 1. Log-linearity of the production function (330); 2. Constant weights on the variances; 3. Log-linearity of the stochastic processes (331) -(333); 4. Constant conditional variances of the shocks (σ's).

Literature on exogenous volatility relaxes assumption 4 by assuming σ's are time-varying, and subject to uncertainty shocks.

CES production function. Under CES technology given by

$$y_t = a_t + \frac{\nu}{\nu-1} \ln\left[\alpha \exp\left(\frac{\nu-1}{\nu} k_t\right) + (1-\alpha)\exp\left(\frac{\nu-1}{\nu} n_t\right)\right], \tag{335}$$

where ν is the elasticity of substitution. The approximation for the conditional variance (334) becomes

$$U_{t,t+1}^{y} \approx \sigma_a^2 + \left[f_k\left(E_t k_{t+1}, E_t n_{t+1}\right)\right]^2 \sigma_k^2 + \left[f_n\left(E_t k_{t+1}, E_t n_{t+1}\right)\right]^2 \sigma_n^2 \tag{336}$$

where $f_k\left(E_t k_{t+1}, E_t n_{t+1}\right)$ denotes the partial derivative of production function w.r.t. k_t.

Even absent of exogenous variation in σ's, now uncertainty in output growth is time-varying due to the state-dependent effects of the shocks.

10.8. **Other Paper on Endogenous Uncertainty.**

10.8.1. **Ilut, Kehrig and Schneider (2018, JPE).* This paper provides a unified explanation for countercyclical volatility both in the cross section and in aggregate time series with a concave hiring rules: when facing firm-level shocks to, say, productivity, they respond more to bad shocks than to good shocks.

Mechanism. We assume that firms' signals (or actual profit) can be decomposed into a common component a and an idiosyncratic component ε (i.i.d.):

$$s_i = a + \varepsilon_i \tag{337}$$

Firms respond to signals about future profitability by changing employment following the same decision rule:

$$n_i = f(s_i) \tag{338}$$

where $f()$ is an increasing and concave function, capturing asymmetric adjustment: firms respond less to good signals than to bad signals.

We can refer to high values of a as representing "good times". Denote the CDFs of employment growth and signals, conditional on a, as $G_n(n|a)$ and $G_s(s|a)$ respectively. The figure below illustrates the mechanism: Bad news (distribution shown in dark gray curve of lower panel) generates larger aggregate responses: the change in aggregate growth in response to bad times (the difference between the horizontal black and dark gray solid lines) is larger than the change in growth in response to bad times (the difference between the horizontal light gray and black lines).

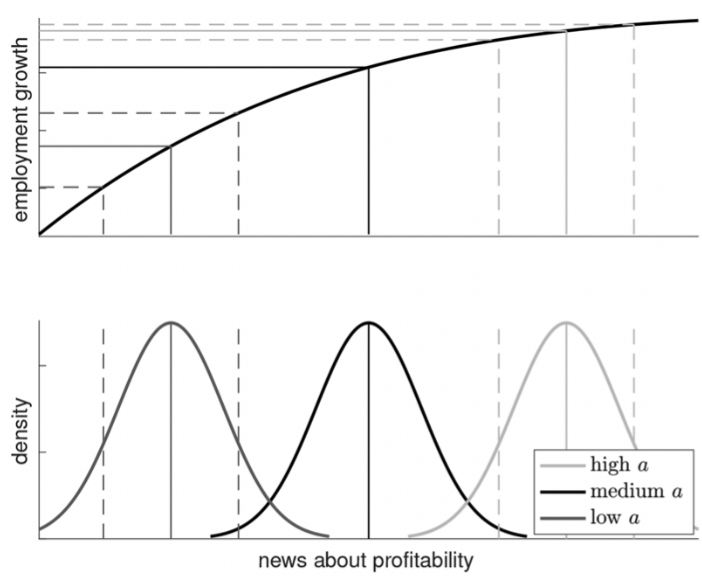


FIGURE 34. Employment growth and signals.

The mechanism can be formalized as the following proposition: For any two aggregate shock realizations $a < a'$,

1. the sensitivity of the aggregate action with respect to the aggregate shock is higher at a :

$$\left.\frac{d}{d\tilde{a}} E[n \mid \tilde{a}]\right|_{\tilde{a}=a} > \left.\frac{d}{d\tilde{a}} E[n \mid \tilde{a}]\right|_{\tilde{a}=a'};$$

2. the cross-sectional variance is higher at a :

$$\operatorname{var}(n \mid a) > \operatorname{var}(n \mid a')$$

10.8.2. **Plante, Richter,& Throckmorton (2018, EJ).* This paper contends that the zero lower bound (ZLB) on the federal funds rate contributed to negative correlation between uncertainty and real GDP growth observed in the data since 2008, and estimates a New Keynesian model with a ZLB constraint to test the theory.

Mechanism A discount factor shock causes households to postpone consumption, which reduces real GDP growth on impact. ZLB generates uncertainty because it creates a kink in the policy functions. When the nominal interest rate is far from its ZLB, the drop in real GDP growth is damped by the monetary policy response, implying little change in uncertainty perceived by the household. When the ZLB binds, the central bank cannot respond by lowering its policy rate, which leads to larger declines in real GDP and higher uncertainty.

11. External Finance Cycles in Macroeconomics

11.1. *** Jermann and Quadrini (2012, AER).** The paper documents that aggregate flow of firm debt financing is procyclical, while equity financing is countercyclical. The paper provides a model with explicit role of debt and equity financing subject to financial shocks to generates observed real and financial dynamics.

Setting. The baseline model consists of a representative firm and household.

Firms. The firm is endowed with Cobb-Douglas technology

$$y_t = z_t k_t^{\theta} n_t^{1-\theta}$$

s.t. capital law of motion

$$k_{t+1} = i_t + (1-\delta)k_t$$

and an intertemporal budget constraint

$$b_t + w_t n_t + k_{t+1} + d_t = (1-\delta)k_t + y_t + \frac{b_{t+1}}{R_t}$$

The firm enters each period with predetermined capital k_t and debt repayment b_t, hires labor n_t, and chooses investment i_t, equity payout (dividend d_t) and borrowing b_{t+1} before production. Firms raise funds with an intra-temporal loan, l_t, to finance working capital.

$$l_t = w_t n_t + i_t + d_t + b_t - \frac{b_{t+1}}{R_t}$$

The ability to borrow (intra- and inter-temporally) is bounded by the limited enforceability of debt contracts as firms can default on their obligations. This friction on debt finance gives rise to a borrowing constraint which is assumed to be binding [36]

$$\xi_t(k_{t+1} - \frac{b_{t+1}}{R_t}) \geq l_t$$

Equity finance is also subject to an adjustment cost, such that actual cost of equity payout is

$$\varphi(d_t) = d_t + \kappa(d_t - \bar{d})^2$$

where $\kappa \geq 0$ and $\bar{d}$ is steady state equity payout.

Formally, the firm's problem is to solve the following Bellman equation:

$$V(\mathbf{s}_t; k_t, b_t) = \max_{d_t, n_t, k_{t+1}, b_{t+1}} d_t + E_t m_{t+1} V(\mathbf{s}_{t+1}; k_{t+1}, b_{t+1}) \tag{339}$$

s.t.

$$(1-\delta)k_t + y_t - w_t n_t + \frac{b_{t+1}}{R_t} = b_t + \varphi(d_t) + k_{t+1} \tag{340}$$

and

$$\xi_t(k_{t+1} - \frac{b_{t+1}}{R_t}) \geq y_t \tag{341}$$

where $\mathbf{s}_t$ summarize aggregate state, and m_{t+1} is a stochastic discount factor consistent with household problem below.

Household. The representative household maximize

$$\max_{c_t, n_t, s_{t+1}} E_0 \sum_{0}^{\infty} \beta^t \{\log c_t + \alpha \log(1-n_t)\} \tag{342}$$

[36] In the paper binding enforcement constraint is micro-founded by a tax benefit.

subject to a borrowing constraint:

$$w_t n_t + b_t + s_t(d_t + p_t) = \frac{b_t}{R_t} + s_{t+1} p_t + c_t \tag{343}$$

where p_t is market price of stock.

Impact of Financial Shock.

- equilibrium with $\kappa = 0$:
  - frictionless in equity finance
  - investment and labor decisions are unconstrained
  - economy resembles standard RBC
  - financial shocks on ξ_t has no real impact
- equilibrium with $\kappa > 0$:

$$f_n(z, k, n) = w \left(\frac{1}{1 - \mu \varphi_d(d)} \right)$$

  - $\xi_t \downarrow \rightarrow \mu \uparrow \rightarrow \left(\frac{1}{1-\mu\varphi_d(d)} \right) \uparrow$
  - if firms want to keep the same scale $\rightarrow$ raise the equity finance. $\rightarrow$ costly $\rightarrow$ equity payout $\downarrow$ + input of labor $\downarrow$
  - financial shocks has real effect

12. Capital Reallocation and Misallocation

12.1. *** Jovanovic and Rousseau (2002, AER).** This paper studies trade-off between investment and merger in a partial equilibrium model. The paper predicts a "high-buy-low" pattern in merger and acquisition, such that resources are reallocated to firms with higher efficiency.

Introduction.

- Two types of capital accumulation
  - Direct investment (internal accumulation, new capital)
  - M&As (external acquisition, used capital[37])
- Nature of cost
  - Investment: high marginal adjustment cost, low fixed cost
  - M&As: high fixed cost, low marginal adjustment cost
- Q-theory of (direct) investment
  - Higher Q, higher investment rate
  - Does this apply to M&A investment?
- Highlights of this paper
  - Q-theory applies to M&A investment as well, and
  - M&As respond to Q more than direct investment
  - M&As waves are reallocative waves
  - High-Q firms by low-Q firms
  - Capital flows to better projects and management

Settings.

- Production function:

$$f(K_t, N_t) = z_t K_t^{\alpha} N_t^{1-\alpha} \tag{344}$$

- z: idiosyncratic technology
  - Markov chain process: $Pr(z_{t+1} = z' | z_t = z) = F(z', z)$
  - positive correlated: $F_2(z', z) < 0$
- K: capital
  - law of motion:

$$K' = (1 - \delta)K + X + Y \tag{345}$$

  - new or disassembled capital investment (X): price = 1
  - used capital market - salvage : price = s
  - used capital market- M&As: price = q
  - used capital investment (Y): q = s (no arbitrage condition[38])
  - investment rate (intensity): $x = X/K$
  - merger rate (intensity): $y = Y/K$
  - law of motion:

$$K' = (1 - \delta + x + y)K \tag{346}$$

[37]In reality, there are two distinct used capital market, one for equipment transactions only, the other for transactions with restructures. In the paper doesn't differentiate between these two markets, as they move together.

[38]This is not generally true, adding credit frictions may distort this result.

Bellman Equation:

$$V(z,K) = \max_{x,y,N} zK^{\alpha}N^{1-\alpha} - wN - C(x,y)K - xK - qyK + \frac{1}{1+r}E_z\{V(z',K')\} \tag{347}$$

- adjustment cost

$$C(x,y) = \begin{cases} c(x,y)+\phi, & if \quad y \neq 0 \\ c(x,y)+0, & if \quad y = 0 \end{cases} \tag{348}$$

- static labor choice:
  - w taken as given:

$$(1-\alpha)z(K/N)^{\alpha} = w \tag{349}$$

  - labor demand linear in capital

$$N = [\frac{w}{(1-\alpha)z}]^{1/\alpha}K \tag{350}$$

- AK technology:

$$A \equiv zK^{\alpha}N^{1-\alpha} - wN = a(z)K \tag{351}$$

- Bellman Equation:

$$V(z,K) = \max_{x,y} a(z)K - C(x,y)K - xK - qyK + \frac{1}{1+r}E_z\{V(z',K')\} \tag{352}$$

- Tobin's Q ($\equiv \frac{V(z,K)}{K}$):

$$Q(z) = \max_{x,y} a(z) - C(x,y) - x - qy + \frac{1-\delta+x+y}{1+r}\{Q^*(z)\} \tag{353}$$

  where

$$Q^*(z) = E(\frac{V(z',K')}{K'}|z) = \int \max\{q, Q(z')\}F(z',z) \tag{354}$$

Without fixed cost:

$$Q(z) = \max_{x,y} a(z) - C(x,y) - x - qy + \frac{1-\delta+x+y}{1+r}Q^*(z) \tag{355}$$

- f.o.c (at interior maximum)
  - $c_1(x,y) + 1 = \frac{1}{1+r}Q^*(z)$
  - $c_2(x,y) + q = \frac{1}{1+r}Q^*(z)$
- $Q^*(z)$ increases with z
  - z is positively auto-correlated
  - high-z firms will grow faster
- without fixed cost,
  - all firms: x & y
  - no difference b/w large and small firms
  - no optimal firm size (only optimal growth)

$$Q(z) = \max_{x;y} a(z) - c(x,y) - \phi 1_{\{y>0\}} - x - qy + \frac{1-\delta+x+y}{1+r}Q^*(z) \tag{356}$$

- gross investment rate: i=x+y
- with fixed cost,
  - low-i firms: x (avoid fixed cost ϕ: y=0)
  - high-i firms: x & y

- participation in merger
  - cut-off value of i^* [39]

$$i^* + c(i^*, 0) = \phi + \min_y\{(i^* - y) + qy + c(i^* - y, y)\} \tag{357}$$

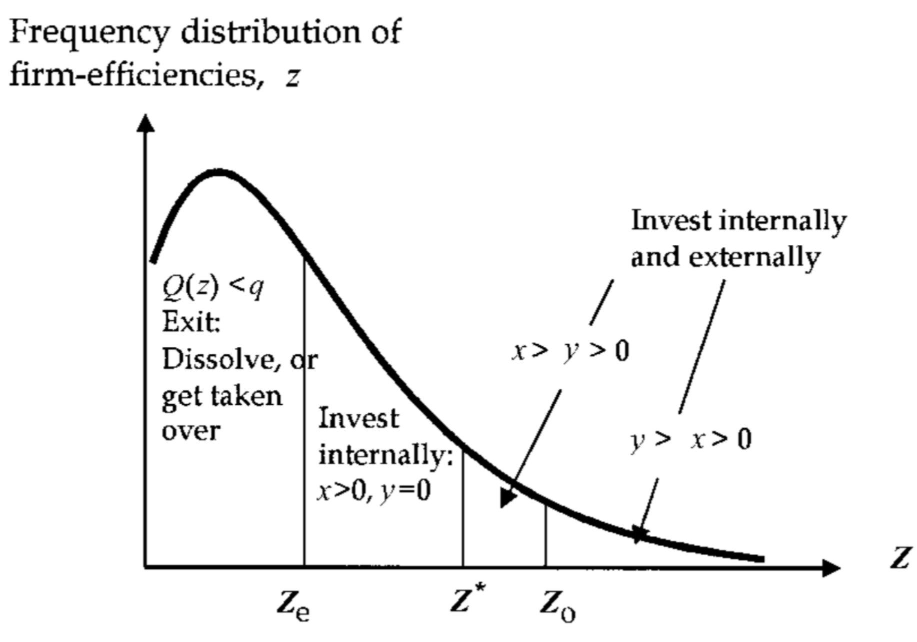
  - cut-off value of z^*
- intensity of merger (vs. investment)
  - cut-off value of i_o
  - cut-off value of z_o

Continuation Problem:

- If continue,
  - value of K: Q(z)K
  - internal value of capital
- If quit
  - value of K: qK
  - outside value of capital
- Cut-off value of z_e: participation in production
  - $Q(z_e) = q$

Cut-off values of z:

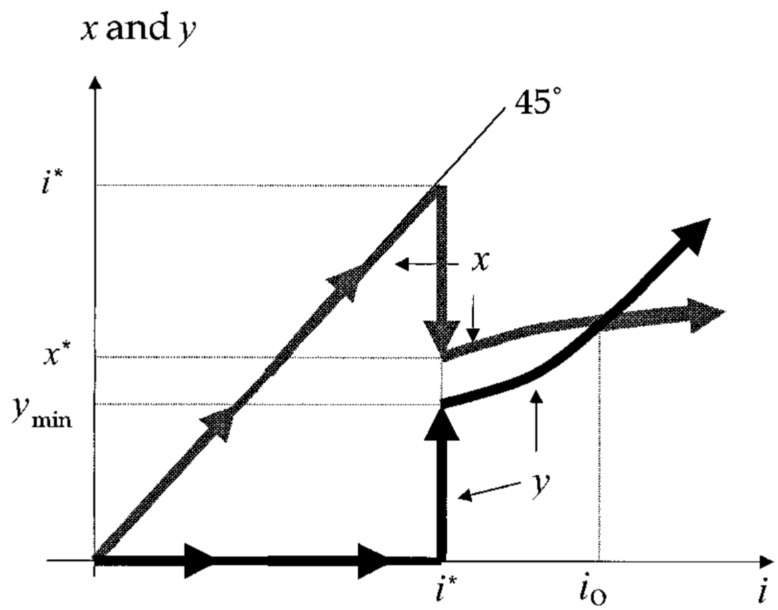
$$z_e < z^* < z_o \tag{358}$$



Distribution of firms.

Figure: The Cut-off Values of z ($z_e < z^* < z_o$)

[39]existence: LHS increases with i, while RHS decreases with i.



Investment Strategy.

Figure: The Expansion Path of x and y (Model)

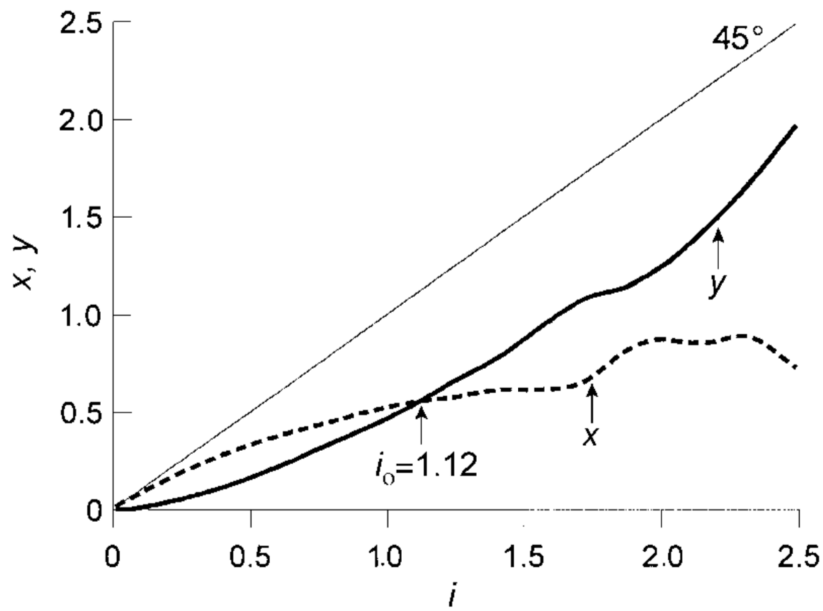


Figure: The Expansion Path of x and y (Data: 1971-2001)

Discussion.

- A seminal work on M&As:
  - high-buys-low pattern
    - “q-theory of mergers”
    - resources transferred from low to high productivity firms

    * merger waves as reallocation waves
    * challenged by Rhodes-Kropf and Robinson (2008): “like-by-like”
  - merger more sensitive to firm’s q than direct investment
    * by a factor of 2.6
- Limitation:
  - no size effect
  - constant resale price of capital
  - a-cyclical merger
  - no general equilibrium effect

12.2. **Eisfeldt and Rampini (2006, JME).** This paper is the seminal work on capital reallocation over business cycles. The paper documents a well-known puzzle that capital reallocation is procyclical while beneifit to it seems to be countercyclical. The authors argue that there must exist some countercyclical friction impeding efficient reallocation, i.e. as in Jovanovic and Rousseau (2002).

12.3. **Hsieh and Klenow (2009, QJE).** This paper is a workhorse model in the field of capital misallocation. The paper documents sizable gaps in marginal products of labor and capital across plants within narrowly defined industries in China and India compared with the United States.

12.4. *** David and Venkateswaran (2019, AER).** This paper explores the sources of arpk dispersion within a unified framework and thus provide a more robust decomposition, including technological/informational frictions and a rich class of firm-specific factors. The identification strategy in this paper is of independent theoretical merit.

Baseline Model. The model features a household sector, intermediate and final good producers.
final goods producer.

- production technology of final product:

$$Y_t = (\int \hat{A}_{it} Y_{it}^{\frac{\theta-1}{\theta}} di)^{\frac{\theta}{\theta-1}} \tag{359}$$

- optimization problem (w. output price normalized to 1):

$$\max_{Y_{it}} Y_t - P_{it}Y_{it} = (\int \hat{A}_{it} Y_{it}^{\frac{\theta-1}{\theta}} di)^{\frac{\theta}{\theta-1}} - P_{it}Y_{it} \tag{360}$$

- first order condition w.r.t. Y_{it}:

$$[(\int \hat{A}_{it} Y_{it}^{\frac{\theta-1}{\theta}} di)^{\frac{\theta}{\theta-1}}]^{\frac{1}{\theta}} \hat{A}_{it} Y_{it}^{\frac{-1}{\theta}} = P_{it}$$

  or

$$Y_t^{\frac{1}{\theta}} \hat{A}_{it} Y_{it}^{-\frac{1}{\theta}} = P_{it} \tag{361}$$

intermediate goods producer.

- production technology of intermediate product:

$$Y_{it} = K_{it}^{\hat{\alpha_1}} N_{it}^{\hat{\alpha_2}} \tag{362}$$

- revenue function:

$$P_{it}Y_{it} = Y_t^{\frac{1}{\theta}} \hat{A}_{it} Y_{it}^{\frac{\theta-1}{\theta}} = Y_t^{\frac{1}{\theta}} \hat{A}_{it} K_{it}^{\hat{\alpha_1}\frac{\theta-1}{\theta}} N_{it}^{\hat{\alpha_2}\frac{\theta-1}{\theta}} \equiv Y_t^{\frac{1}{\theta}} \hat{A}_{it} K_{it}^{\alpha_1} N_{it}^{\alpha_2} \tag{363}$$

  where $\alpha_1 \equiv \hat{\alpha_1}\frac{\theta-1}{\theta}$ and $\alpha_2 \equiv \hat{\alpha_2}\frac{\theta-1}{\theta}$ can be interpreted as either firm-specific demand factor, or idiosyncratic productivity.
- value function $V(K_{it}, I_{it})$:

$$\begin{aligned} \max_{K_{it+1}, N_{it}} E_{it}\{Y_t^{\frac{1}{\theta}} \hat{A}_{it} K_{it}^{\alpha_1} N_{it}^{\alpha_2} - W_t N_{it} - T_{it+1}^K K_{it+1}(1-\beta(1-\delta)) - \Phi(K_{it+1}, K_{it})\} \\ +\beta E_{it} V(K_{it+1}, I_{it+1}) \end{aligned} \tag{364}$$

  where N_{it} is implied from F.O.C. of a static labor choice problem:

$$\alpha_2 Y_t^{\frac{1}{\theta}} \hat{A}_{it} K_{it}^{\alpha_1} N_{it}^{\alpha_2 - 1} = W_t \Rightarrow N_{it} = W_t^{\frac{1}{\alpha_2 - 1}} [\alpha_2 Y_t^{\frac{1}{\theta}} \hat{A}_{it} K_{it}^{\alpha_1}]^{\frac{1}{1-\alpha_2}} \tag{365}$$

  and $\Phi(K_{it+1}, K_{it})$ is capital adjustment cost in a quadratic form:

$$\Phi(K_{it+1}, K_{it}) = \frac{\hat{\xi}}{2}(\frac{K_{it+1}}{K_{it}} - (1-\delta))^2 K_{it} \tag{366}$$

- value function $V(K_{it}, I_{it})$ rewritten as [40]:

$$\max_{K_{it+1}} E_{it}\{G_t A_{it} K_{it}^{\alpha} - T_{it+1}^{K} K_{it+1}(1-\beta(1-\delta)) - \Phi(K_{it+1}, K_{it})\} + \beta E_{it} V(K_{it+1}, I_{it+1}) \quad (367)$$

  where $\alpha = \frac{\alpha_1}{1-\alpha_2}$, $G_t \equiv (1-\alpha_2)\alpha_2^{\frac{\alpha_2}{1-\alpha_2}} Y_t^{\frac{1}{\theta}(\frac{1}{1-\alpha_2})} W_t^{\frac{-\alpha_2}{1-\alpha_2}}$ is an aggregate variable, and $A_{it} \equiv \hat{A}_{it}^{\frac{1}{1-\alpha_2}}$ is *adjusted* idiosyncratic productivity.

equilibrium and solution method.

- the *stationary equilibrium* is comprised of (i) individual value functions and policy functions: $V(K_{it}, I_{it})$, $N(K_{it}, I_{it})$, $K_{it+1}(K_{it}, I_{it})$, (ii) aggregate variable W_t, and (iii) aggregate joint distribution of (K_{it}, I_{it}) that are consistent with stochastic processes and market clearing conditions in subsequent sections.
- perturbation method to solve the model:
  - log-linearize firm's F.O.C.s around $A_{it} = \bar{A}$ and $T_{it}^{K} = 1$
- F.O.C. w.r.t. K_{it+1} of value function:

$$E_{it}\{T_{it+1}^{K}(1-\beta(1\delta)) + \Phi_1(K_{it+1}, K_{it})\} = \beta E_{it} V_1(K_{it+1}, I_{it+1}) \quad (368)$$

- envelope theorem:

$$V_1(K_{it}, I_{it}) = \alpha G A_{it} K_{it}^{\alpha-1} - \Phi_2(K_{it+1}, K_{it}) \quad (369)$$

- combining above two equations gives Euler equation:

$$E_{it}\{T_{it+1}^{K}(1-\beta(1\delta)) + \Phi_1(K_{it+1}, K_{it})\} = \beta E_{it}\{\alpha G A_{it+1} K_{it+1}^{\alpha-1} - \Phi_2(K_{it+2}, K_{it+1})\}$$

  or

$$E_{it}\{T_{it+1}^{K}(1-\beta(1\delta)) + \hat{\xi}[\frac{K_{it+1}}{K_{it}} - (1-\delta)]\} = \beta E_{it}\{\alpha G A_{it+1} K_{it+1}^{\alpha-1} - \frac{\hat{\xi}}{2}(1-\delta)^2 + \frac{\hat{\xi}}{2}(\frac{K_{it+1}}{K_{it}})^2\} \quad (370)$$

- log-linearized Euler equation:

$$k_{it+1}[\xi(1+\beta) + 1 - \alpha] = E_{it}[a_{it+1} + \tau_{it+1}] + \beta\xi E_{it} k_{it+2} + \xi k_{it} \quad (371)$$

  where $\xi = \frac{\hat{\xi}}{\beta\alpha\bar{G}\bar{A}\bar{K}^{\alpha-1}}$ and $\tau_{it+1} = -\frac{1-\beta(1-\delta)}{\beta\alpha\bar{G}\bar{A}\bar{K}^{\alpha-1}}\tau_{it+1}^{K}$.

---

[40]

$$\begin{aligned}
V(K_{it}, I_{it}) = \max_{K_{it+1}} & E_{it}\{Y_t^{\frac{1}{\theta}} \hat{A}_{it} K_{it}^{\alpha_1} W_t^{\frac{-\alpha_2}{1-\alpha_2}} [\alpha_2 Y_t^{\frac{1}{\theta}} \hat{A}_{it} K_{it}^{\alpha_1}]^{\frac{\alpha_2}{1-\alpha_2}} - W_t[W_t^{\frac{-1}{1-\alpha_2}} [\alpha_2 Y_t^{\frac{1}{\theta}} \hat{A}_{it} K_{it}^{\alpha_1}]^{\frac{1}{1-\alpha_2}}] \\
& - T_{it+1}^{K} K_{it+1}(1-\beta(1-\delta)) - \Phi(K_{it+1}, K_{it})\} + \beta E_{it} V(K_{it+1}, I_{it+1}) \\
= \max_{K_{it+1}} & E_{it}\{\alpha_2^{\frac{\alpha_2}{1-\alpha_2}} Y_t^{\frac{1}{\theta}(1+\frac{\alpha_2}{1-\alpha_2})} \hat{A}_{it}^{1+\frac{\alpha_2}{1-\alpha_2}} W_t^{\frac{-\alpha_2}{1-\alpha_2}} K_{it}^{\alpha_1} K_{it}^{\alpha_1 \frac{\alpha_2}{1-\alpha_2}} - W_t^{1-\frac{1}{1-\alpha_2}} [\alpha_2^{\frac{1}{1-\alpha_2}} Y_t^{\frac{1}{\theta}\frac{1}{1-\alpha_2}} \hat{A}_{it}^{\frac{1}{1-\alpha_2}} K_{it}^{\alpha_1 \frac{1}{1-\alpha_2}}] \\
& - T_{it+1}^{K} K_{it+1}(1-\beta(1-\delta)) - \Phi(K_{it+1}, K_{it})\} + \beta E_{it} V(K_{it+1}, I_{it+1}) \\
= \max_{K_{it+1}} & E_{it}\{\alpha_2^{\frac{\alpha_2}{1-\alpha_2}} Y_t^{\frac{1}{\theta}(\frac{1}{1-\alpha_2})} W_t^{\frac{-\alpha_2}{1-\alpha_2}} \hat{A}_{it}^{\frac{1}{1-\alpha_2}} K_{it}^{\frac{\alpha_1}{1-\alpha_2}} - \alpha_2^{\frac{1}{1-\alpha_2}} Y_t^{\frac{1}{\theta}\frac{1}{1-\alpha_2}} W_t^{\frac{-\alpha_2}{1-\alpha_2}} \hat{A}_{it}^{\frac{1}{1-\alpha_2}} K_{it}^{\frac{\alpha_1}{1-\alpha_2}} \\
& - T_{it+1}^{K} K_{it+1}(1-\beta(1-\delta)) - \Phi(K_{it+1}, K_{it})\} + \beta E_{it} V(K_{it+1}, I_{it+1}) \\
= \max_{K_{it+1}} & E_{it}\{(1-\alpha_2)\alpha_2^{\frac{\alpha_2}{1-\alpha_2}} Y_t^{\frac{1}{\theta}(\frac{1}{1-\alpha_2})} W_t^{\frac{-\alpha_2}{1-\alpha_2}} \hat{A}_{it}^{\frac{1}{1-\alpha_2}} K_{it}^{\frac{\alpha_1}{1-\alpha_2}} - T_{it+1}^{K} K_{it+1}(1-\beta(1-\delta)) - \Phi(K_{it+1}, K_{it})\} \\
& + \beta E_{it} V(K_{it+1}, I_{it+1}) \\
\equiv \max_{K_{it+1}} & E_{it}\{G_t A_{it} K_{it}^{\alpha} - T_{it+1}^{K} K_{it+1}(1-\beta(1-\delta)) - \Phi(K_{it+1}, K_{it})\} + \beta E_{it} V(K_{it+1}, I_{it+1})
\end{aligned}$$

- implied investment policy function (conjecture and verify):

$$k_{it+2} = \psi_1 k_{it+1} + \psi_2(1+\gamma)E_{it+1}a_{it+2} + \psi_3\varepsilon_{it+2} + \psi_4\chi_i \tag{372}$$

where

$$\begin{aligned} \xi(\beta\psi_1^2+1) &= \psi_1[\xi(1+\beta+1-\alpha)] \\ \psi_2 &= \frac{\psi_1}{\xi(1-\beta\rho\psi_1^2)} \\ \psi_3 &= \frac{\psi_1}{\xi} \\ \psi_4 &= \frac{1-\psi_1}{1-\alpha} \end{aligned}$$

stochastic processes.

- *adjusted* productivity, A_{it} follows an AR(1) process in logs (a_{it}):

$$a_{it} = \rho a_{it-1} + \mu_{it}, \quad \mu_{it} \sim N(0,\sigma_\mu^2) \tag{373}$$

- *re-scaled* distortion parameter in log take the form:

$$\tau_{it} = \gamma a_{it} + \varepsilon_{it} + \chi_i, \quad \varepsilon_{it} \sim N(0,\sigma_\varepsilon^2), \chi_i \sim N(0,\sigma_\chi^2) \tag{374}$$

labor market clearing condition.

- aggregate labor supply: N_t
- aggregate labor demand: from equation (7)

$$\int N_{it}di = \int W_t^{\frac{1}{\alpha_2-1}}[\alpha_2 Y_t^{\frac{1}{\theta}}\hat{A}_{it}K_{it}^{\alpha_1}]^{\frac{1}{1-\alpha_2}}di = (\frac{\alpha_2 Y_t^{\frac{1}{\theta}}}{W_t})^{\frac{1}{1-\alpha_2}}\int A_{it}K_{it}^{\alpha}di \tag{375}$$

where the last equation utilizes notation $\alpha = \frac{\alpha_1}{1-\alpha_2}$ and $A_{it} \equiv \hat{A}_{it}^{\frac{1}{1-\alpha_2}}$.

- labor market clearing condition:

$$(\frac{\alpha_2}{W_t})^{\frac{1}{1-\alpha_2}} Y_t^{\frac{1}{\theta}(\frac{1}{1-\alpha_2})}\int A_{it}K_{it}^{\alpha}di = N_t \tag{376}$$

capital market clearing condition.

- The last equation implies that

$$\begin{aligned} P_{it}Y_{it} &= \alpha_2^{\frac{\alpha_2}{1-\alpha_2}} Y_t^{\frac{1}{\theta}(\frac{1}{1-\alpha_2})} W_t^{\frac{-\alpha_2}{1-\alpha_2}} A_{it}K_{it}^{\alpha} = (\frac{\alpha_2}{W_t})^{\frac{\alpha_2}{1-\alpha_2}} Y_t^{\frac{1}{\theta}(\frac{1}{1-\alpha_2})} W_t^{\frac{-\alpha_2}{1-\alpha_2}} A_{it}K_{it}^{\alpha} \\ &= [\frac{N_t}{Y_t^{\frac{1}{\theta}(\frac{1}{1-\alpha_2})}\int A_{it}K_{it}^{\alpha}di}]^{\alpha_2} Y_t^{\frac{1}{\theta}(\frac{1}{1-\alpha_2})} W_t^{\frac{-\alpha_2}{1-\alpha_2}} A_{it}K_{it}^{\alpha} = Y_t^{\frac{1}{\theta}}\frac{A_{it}K_{it}^{\alpha}}{[\int A_{it}K_{it}^{\alpha}di]^{\alpha_2}} N_t^{\alpha_2} \end{aligned} \tag{377}$$

- average return to capital ($ARPK_{it}$):

$$ARPK_{it} \equiv \frac{P_{it}Y_{it}}{K_{it}} = \frac{A_{it}K_{it}^{\alpha-1}}{[\int A_{it}K_{it}^{\alpha}di]^{\alpha_2}} Y_t^{\frac{1}{\theta}} N_t^{\alpha_2} \tag{378}$$

which implies

$$K_{it} = Y_t^{\frac{1}{\theta}\frac{1}{1-\alpha}}[\frac{N_t}{\int A_{it}K_{it}^{\alpha}di}]^{\frac{\alpha_2}{1-\alpha}}[\frac{A_{it}}{ARPK_{it}}]^{\frac{1}{1-\alpha}} \tag{379}$$

- aggregate capital demand:

$$\int K_{it}di = Y_t^{\frac{1}{\theta}\frac{1}{1-\alpha}}[\frac{N_t}{\int A_{it}K_{it}^{\alpha}di}]^{\frac{\alpha_2}{1-\alpha}}\int[\frac{A_{it}}{ARPK_{it}}]^{\frac{1}{1-\alpha}}di \tag{380}$$

- aggregate capital supply: K_t
- capital market clearing condition:

$$Y_t^{\frac{1}{\theta}\frac{1}{1-\alpha}}[\frac{N_t}{\int A_{it}K_{it}^{\alpha}di}]^{\frac{\alpha_2}{1-\alpha}}\int[\frac{A_{it}}{ARPK_{it}}]^{\frac{1}{1-\alpha}}di = K_t \tag{381}$$

aggregation.

- K_{it} is implied from equation (21) and equation (23):

$$K_{it} = \frac{K_t}{\int[\frac{A_{it}}{ARPK_{it}}]^{\frac{1}{1-\alpha}}di}[\frac{A_{it}}{ARPK_{it}}]^{\frac{1}{1-\alpha}} = \frac{A_{it}^{\frac{1}{1-\alpha}}ARPK_{it}^{\frac{-1}{1-\alpha}}}{\int A_{it}^{\frac{1}{1-\alpha}}ARPK_{it}^{\frac{-1}{1-\alpha}}di}K_t \tag{382}$$

- revenue function of equation (19) re-written:

$$\begin{aligned} P_{it}Y_{it} &= Y_t^{\frac{1}{\theta}}\frac{A_{it}K_{it}^{\alpha}}{[\int A_{it}K_{it}^{\alpha}di]^{\alpha_2}}N_t^{\alpha_2} = Y_t^{\frac{1}{\theta}}\frac{A_{it}[\frac{A_{it}^{\frac{1}{1-\alpha}}ARPK_{it}^{\frac{-1}{1-\alpha}}}{\int A_{it}^{\frac{1}{1-\alpha}}ARPK_{it}^{\frac{-1}{1-\alpha}}di}K_t]^{\alpha}}{\{\int A_{it}[\frac{A_{it}^{\frac{1}{1-\alpha}}ARPK_{it}^{\frac{-1}{1-\alpha}}}{\int A_{it}^{\frac{1}{1-\alpha}}ARPK_{it}^{\frac{-1}{1-\alpha}}di}K_t]^{\alpha}di\}^{\alpha_2}}N_t^{\alpha_2} \\ &= \frac{\frac{A_{it}^{\frac{1}{1-\alpha}}ARPK_{it}^{\frac{-\alpha}{1-\alpha}}}{[\int A_{it}^{\frac{1}{1-\alpha}}ARPK_{it}^{\frac{-1}{1-\alpha}}di]^{\alpha}}K_t^{\alpha}}{\{K_t^{\alpha}\int A_{it}[\frac{A_{it}^{\frac{1}{1-\alpha}}ARPK_{it}^{\frac{-1}{1-\alpha}}}{\int A_{it}^{\frac{1}{1-\alpha}}ARPK_{it}^{\frac{-1}{1-\alpha}}di}]^{\alpha}di\}^{\alpha_2}}Y_t^{\frac{1}{\theta}}N_t^{\alpha_2} = \frac{\frac{A_{it}^{\frac{1}{1-\alpha}}ARPK_{it}^{\frac{-\alpha}{1-\alpha}}}{[\int A_{it}^{\frac{1}{1-\alpha}}ARPK_{it}^{\frac{-1}{1-\alpha}}di]^{\alpha}}}{\{\frac{\int A_{it}^{\frac{1}{1-\alpha}}ARPK_{it}^{\frac{-\alpha}{1-\alpha}}di}{[\int A_{it}^{\frac{1}{1-\alpha}}ARPK_{it}^{\frac{-1}{1-\alpha}}di]^{\alpha}}\}^{\alpha_2}}Y_t^{\frac{1}{\theta}}K_t^{\alpha_1}N_t^{\alpha_2} \end{aligned} \tag{383}$$

where last equation utilizes the notation: $\alpha \equiv \frac{\alpha_1}{1-\alpha_2}$ so that $K_t^{\frac{\alpha}{1-\alpha_2}} = K_t^{\alpha_1}$

- from last equation:

$$Y_t = \int P_{it}Y_{it}di \equiv A_tY_t^{\frac{1}{\theta}}K_t^{\alpha_1}N_t^{\alpha_2}$$

so that

$$Y_t = A_t^{\frac{\theta}{\theta-1}}K_t^{\hat{\alpha}_1}N_t^{\hat{\alpha}_2} \tag{384}$$

where we utilize the notation: $\alpha_1 = \frac{\theta-1}{\theta}\hat{\alpha}_1$, $\alpha_2 = \frac{\theta-1}{\theta}\hat{\alpha}_2$ and

$$A_t \equiv \int \frac{\frac{A_{it}^{\frac{1}{1-\alpha}}ARPK_{it}^{\frac{-\alpha}{1-\alpha}}}{[\int A_{it}^{\frac{1}{1-\alpha}}ARPK_{it}^{\frac{-1}{1-\alpha}}di]^{\alpha}}}{\{\frac{\int A_{it}^{\frac{1}{1-\alpha}}ARPK_{it}^{\frac{-\alpha}{1-\alpha}}di}{[\int A_{it}^{\frac{1}{1-\alpha}}ARPK_{it}^{\frac{-1}{1-\alpha}}di]^{\alpha}}\}^{\alpha_2}}di = [\frac{\int A_{it}^{\frac{1}{1-\alpha}}ARPK_{it}^{\frac{-\alpha}{1-\alpha}}di}{(\int A_{it}^{\frac{1}{1-\alpha}}ARPK_{it}^{\frac{-1}{1-\alpha}}di)^{\alpha}}]^{1-\alpha_2} \tag{385}$$

- take log of equation (27) (suppressing time script for some aggregate variables):

$$\begin{aligned} a_t \equiv \log(A_t) &= (1-\alpha_2)[\log(\int A_{it}^{\frac{1}{1-\alpha}} ARPK_{it}^{\frac{-\alpha}{1-\alpha}} dj) - \alpha \log(\int A_{it}^{\frac{1}{1-\alpha}} ARPK_{it}^{\frac{-1}{1-\alpha}} di)] \\ &= (1-\alpha_2)\{\frac{1}{1-\alpha}\bar{a} - \frac{\alpha}{1-\alpha}\bar{arpk} + \frac{1}{2}(\frac{1}{1-\alpha})^2\sigma_a^2 + \frac{1}{2}(\frac{\alpha}{1-\alpha})^2\sigma_{arpk}^2 - \frac{\alpha}{(1-\alpha)^2}\sigma_{arpk,a} \\ &- \alpha[\frac{1}{1-\alpha}\bar{a} - \frac{\alpha}{1-\alpha}\bar{arpk} + \frac{1}{2}\alpha(\frac{1}{1-\alpha})^2\sigma_a^2 + \frac{1}{2}\alpha(\frac{\alpha}{1-\alpha})^2\sigma_{arpk}^2 - \frac{\alpha}{(1-\alpha)^2}\sigma_{arpk,a}]\} \\ &= (1-\alpha_2)[\bar{a} + \frac{1}{2}\frac{1}{1-\alpha}\sigma_a^2 - \frac{1}{2}\frac{\alpha}{1-\alpha}\sigma_{arpk}^2] \end{aligned} \tag{386}$$

- take log of equation (26) (given that $\alpha \equiv \frac{\alpha_1}{1-\alpha_2}$ and $\hat{\alpha}_1 \equiv \frac{\theta}{\theta-1}\alpha_1$):

$$\begin{aligned} y_t &= \frac{\theta}{\theta-1}a_t + \hat{\alpha}_1 k_t + \hat{\alpha}_2 n_t \\ &= \frac{\theta}{\theta-1}(1-\alpha_2)\bar{a} + \frac{\theta}{\theta-1}\frac{1}{2}\frac{(1-\alpha_2)}{1-\alpha}\sigma_a^2 - \frac{\theta}{\theta-1}\frac{1}{2}\frac{\alpha(1-\alpha_2)}{1-\alpha}\sigma_{arpk}^2 + \hat{\alpha}_1 k_t + \hat{\alpha}_2 n_t \\ &= \frac{\theta}{\theta-1}\bar{\hat{a}} + \frac{\theta}{\theta-1}\frac{1}{2}\frac{(1-\alpha_2)}{(1-\alpha)(1-\alpha_2)^2}\sigma_{\hat{a}}^2 - \frac{\theta}{\theta-1}\frac{1}{2}\frac{\alpha(1-\alpha_2)}{1-\alpha}\sigma_{arpk}^2 + \hat{\alpha}_1 k_t + \hat{\alpha}_2 n_t \\ &= \frac{\theta}{\theta-1}\bar{\hat{a}} + \frac{\theta}{\theta-1}\frac{1}{2}\frac{1}{(1-\alpha_2-\alpha_1)}\sigma_{\hat{a}}^2 - \frac{1}{2}(\theta\hat{\alpha}_1 + \hat{\alpha}_2)\hat{\alpha}_1\sigma_{arpk}^2 + \hat{\alpha}_1 k_t + \hat{\alpha}_2 n_t \\ &\equiv a^* - \frac{1}{2}(\theta\hat{\alpha}_1 + \hat{\alpha}_2)\hat{\alpha}_1\sigma_{arpk}^2 + \hat{\alpha}_1 k_t + \hat{\alpha}_2 n_t \end{aligned} \tag{387}$$

  where $k_t \equiv \log(K_t)$, $n_t \equiv \log(N_t)$, $\hat{a} \equiv (1-\alpha_2)a$, and $a^* \equiv \frac{\theta}{\theta-1}\bar{\hat{a}} + \frac{\theta}{\theta-1}\frac{1}{2}\frac{1}{(1-\alpha_2-\alpha_1)}\sigma_{\hat{a}}^2$.
- a^* is aggregate TFP if static capital product ($arpk_{it}$) are equalized across firms and σ_{arpk}^2 is cross-sectional dispersion in log of static average product of capital:

$$ARPK_{it} = \frac{P_{it}Y_{it}}{K_{it}} \Rightarrow \quad arpk_{it} = p_{it}y_{it} - k_{it}$$

- *key insight*:

$$\frac{dy}{d\sigma_{arpk}^2} = \frac{\theta}{\theta-1}\frac{da}{d\sigma_{arpk}^2} = -\frac{1}{2}(\theta\hat{\alpha}_1 + \hat{\alpha}_2)\hat{\alpha}_1 < 0 \tag{388}$$

  Therefore, TFP and aggregate output are monotonically decreases w.r.t. dispersion in capital productivity. A number of factors contribute to dispersion in *arpk*, including adjustment costs, information frictions and other distortions, and once we quantify their contribution to σ_{arpk}^2, we can map directly to output loss in this framework.

Identification.
motivation.

- measuring the contribution of each factor is a challenging task, since all the data moments confound all the factors (i.e., each moment reflects the influence of more than one factor). As a result, there is no one-to-one mapping between individual moments and parameters: to accurately identify the contribution of any factor, we need to explicitly control for the others.
- assumption: preference and technology parameters, including β, α, δ are assumed as given for identification.

- remaining parameters[41]: ξ, V, γ, σ_ε^2 and σ_χ^2.
- methodology: Use a series of elements from covariance matrix of firm-level capital and productivity: (1) autocorrelation of investment $\rho_{i,i_{-1}}$; (2) the variance of investment σ_i^2; (3) correlation of period t investment with lagged innovation in productivity $\rho_{i.a_{-1}}$ and (4) coefficient from regressing $\Delta arpk$ on Δa, denoted as $\rho_{arpk,a}$.

intuition.

- higher adjustment cost parameter ξ:
  - lower σ_i^2;
  - higher $\rho_{i,i_{-1}}$
  - higher $\rho_{i.a_{-1}}$
- higher correlation in distortion γ (more negative):
  - lower σ_i^2;
  - lower $\rho_{i,i_{-1}}$
  - lower $\rho_{i.a_{-1}}$
  - higher $\rho_{arpk,a}$
- higher information friction V (uncertainty):
  - unaffected $\rho_{i,i_{-1}}$
  - higher $\rho_{i.a_{-1}}$
  - higher $\rho_{arpk,a}$
- higher volatility in transitory shock σ_ε^2:
  - lower $\rho_{i,i_{-1}}$
  - unaffected $\rho_{arpk,a}$

pairwise identification.

- σ_i^2 and $\rho_{i,i_{-1}} \Rightarrow \xi$ and γ
- $\rho_{i,a_{-1}}$ and $\rho_{arpk,a} \Rightarrow V$ and γ
- $\rho_{i,i_{-1}}$ and $\rho_{arpk,a} \Rightarrow \sigma_\varepsilon^2$ and γ
- $\rho_{i,i_{-1}}$ and $\rho_{i,a_{-1}} \Rightarrow \xi$ and V

proposition 1.

- start from investment policy function 372

$$k_{it+1} = \psi_1 k_{it} + \psi_2(1+\gamma)E_{it}a_{it+1} + \psi_3\varepsilon_{it+1} + \psi_4\chi_i$$

- utilize log AR(1) process of technology (373) with assumption that $\rho = 1$ to substitute out expectation:

$$k_{it+1} = \psi_1 k_{it} + \psi_2(1+\gamma)[a_{it} + \phi(\mu_{it+1} + e_{it+1})] + \psi_3\varepsilon_{it+1} + \psi_4\chi_i \tag{389}$$

  where $\phi = \frac{V}{\sigma_e^2}$ and $1 - \phi = \frac{V}{\sigma_\mu^2}$.
- first difference last equation gives

$$\begin{aligned} var(\Delta k_{it+1}) &= \psi_1\Delta k_{it} + \psi_2(1+\gamma)[a_{it} - a_{it-1} + \phi(\mu_{it+1} - \mu_{it} + e_{it+1} - e_{it})] + \psi_3[\varepsilon_{it+1} - \varepsilon_{it}] \\ &= \psi_1\Delta k_{it} + \psi_2(1+\gamma)[\mu_{it} + \phi(\mu_{it+1} - \mu_{it} + e_{it+1} - e_{it})] + \psi_3[\varepsilon_{it+1} - \varepsilon_{it}] \\ &= \psi_1\Delta k_{it} + \psi_2(1+\gamma)[(1-\phi)\mu_{it} + \phi\mu_{it+1} + \phi(e_{it+1} - e_{it})] + \psi_3[\varepsilon_{it+1} - \varepsilon_{it}] \end{aligned} \tag{390}$$

[41]In the special case where $\rho = 1$, i.e. productivity follow a random walk, σ_χ^2 cannot be identified. The authors derive analytical solution for four remaining parameters and show intuition behind the mapping between parameters and moments.

- Thus the variance of LHS =

$$
\begin{aligned}
var(\Delta k_{it+1}) &= var(\psi_1 \Delta k_{it}) + var(\psi_2(1+\gamma)[(1-\phi)\mu_{it} + \phi\mu_{it+1}]) \\
&+ var[\psi_2(1+\gamma)\phi(e_{it+1} - e_{it})] + var[\psi_3(\varepsilon_{it+1} - \varepsilon_{it})] \\
&+ 2cov(\psi_1 \Delta k_{it}, \psi_2(1+\gamma)[(1-\phi)\mu_{it} + \phi\mu_{it+1}]) \\
&+ 2cov[\psi_1 \Delta k_{it}, \psi_2(1+\gamma)\phi(e_{it+1} - e_{it})] + 2cov[\psi_1 \Delta k_{it}, \psi_3(\varepsilon_{it+1} - \varepsilon_{it})] \\
&= \psi_1^2 var(\Delta k_{it}) + var[\psi_2(1+\gamma)(1-\phi)\mu_{it}] + var[\psi_2(1+\gamma)\phi\mu_{it+1}] \\
&+ var[\psi_2(1+\gamma)\phi e_{it+1}] + var[\psi_2(1+\gamma)\phi e_{it}] + var[\psi_3\varepsilon_{it+1}] + var[\psi_3\varepsilon_{it}] \\
&+ 2\psi_1\psi_2(1+\gamma)cov(\Delta k_{it}, [(1-\phi)\mu_{it} + \phi\mu_{it+1}]) \\
&+ 2\psi_1\psi_2(1+\gamma)\phi cov[\Delta k_{it}, (e_{it+1} - e_{it})] + 2\psi_1\psi_3 cov[\Delta k_{it}, (\varepsilon_{it+1} - \varepsilon_{it})] \\
&= \psi_1^2 var(\Delta k_{it}) + \psi_2^2(1+\gamma)^2(1-\phi)^2\sigma_\mu^2 + \psi_2^2(1+\gamma)^2\phi^2\sigma_\mu^2 \\
&+ 2\psi_2^2(1+\gamma)^2\phi^2\sigma_e^2 + 2\psi_3^2\sigma_\varepsilon^2 \\
&+ 2\psi_1\psi_2(1+\gamma)cov(\Delta k_{it}, (1-\phi)\mu_{it}) \\
&+ 2\psi_1\psi_2(1+\gamma)\phi cov[\Delta k_{it}, -e_{it}] + 2\psi_1\psi_3 cov[\Delta k_{it}, -\varepsilon_{it}] \\
&= \psi_1^2 var(\Delta k_{it}) + \psi_2^2(1+\gamma)^2(1-\phi)^2\sigma_\mu^2 + \psi_2^2(1+\gamma)^2\phi^2\sigma_\mu^2 \\
&+ 2\psi_2^2(1+\gamma)^2\phi^2\sigma_e^2 + 2\psi_3^2\sigma_\varepsilon^2 \\
&+ 2\psi_1\psi_2(1+\gamma)(1-\phi)cov(\Delta k_{it}, \mu_{it}) \\
&- 2\psi_1\psi_2(1+\gamma)\phi cov[\Delta k_{it}, e_{it}] - 2\psi_1\psi_3 cov[\Delta k_{it}, \varepsilon_{it}]
\end{aligned} \tag{391}
$$

- Rearrange the term:

$$
\begin{aligned}
\sigma_k^2 = \psi_1^2\sigma_k^2 + \psi_2^2(1+\gamma)^2(1-\phi)^2\sigma_\mu^2 + \psi_2^2(1+\gamma)^2\phi^2\sigma_\mu^2 + 2\psi_2^2(1+\gamma)^2\phi^2\sigma_e^2 + 2\psi_3^2\sigma_\varepsilon^2 + \\
2\psi_1\psi_2(1+\gamma)(1-\phi)\psi_2(1+\gamma)\phi\sigma_\mu^2 - 2\psi_1\psi_2(1+\gamma)\phi\psi_2(1+\gamma)\phi\sigma_e^2 - 2\psi_1\psi_3\psi_3\sigma_\varepsilon^2
\end{aligned}
$$

and utilize assumption of stationary moments:

$$
(1-\psi_1^2)\sigma_k^2 = \psi_2^2(1+\gamma)^2[1 - (1-\psi_1)2(\phi+\phi^2)]\sigma_\mu^2 + 2(1-\psi_1)\psi_3^2\sigma_\varepsilon^2
$$

$$
\sigma_k^2 = \frac{(1+\gamma)^2\psi_2^2[1-(1-\psi_1)2(\phi+\phi^2)]\sigma_\mu^2 + 2(1-\psi_1)\psi_3^2\sigma_\varepsilon^2}{1-\psi_1^2} \tag{392}
$$

Data and Measurement: U.S. public firm.
data and sample.

- major data source: Compustat North America;
- complementary data source: NBER-CES Manufacturing Industry Database [42]
- sample period: 1998-2009
- sample size: 34260 firm-year observations [43]
- industry: standard industry classification (SIC) as 4-digit level

[42] for average industry wage to impute a measure of wage bill.

[43] after eliminating duplicate and problematic observations (i.e., firm with multiple observations within a single year, or reporting in foreign currencies), outliers (3% tails of each series), observations with missing data etc.

measurement of variables.

- log of capital stock (k_{it}): log of gross property, plant and equipment "ppegt") [44]
- investment (i_{it}) : $i_{it} = k_{it} - k_{it-1}$
- investment growth (Δi_{it}): $\Delta i_{it} = i_{it} - i_{it-1}$
- log of value-added (va_{it}): estimated as a constant fraction of revenues using a share of intermediates of 0.5 (measured by log of sale in Compustat)
- log of average product of capital ($arpk_{it}$): $arpk_{it} = va_{it} - k_{it}$
- log of firm productivity (a_{it}): $a_{it} = va_{it} - \alpha k_{it}$
- productivity growth (Δa_{it}): $\Delta a_{it} = a_{it} - a_{it-1}$
- labor input(wb_{it}) = wage bill = number of employees * average industry wage rate [45]: $wb_{it} = emp_{it} * wg_{it}$
- total expenses: = sales - operating income [46]
- intermediate expenditure: = total expenses - labor expenses = $sale - oibdp - wb$

parameters: see Table.2.

- annual discount factor (β): 0.95
- annual depreciation rate (δ): 0.1
- elasticity of substitution (θ): 6
- input shares ($\hat{\alpha}_1, \hat{\alpha}_2$): 0.33 and 0.67 for U.S. firms $\rightarrow$ implied $\alpha = 0.62$
- persistence of productivity process (ρ): estimated as shown in section 12.4
- volatility of productivity shocks (σ^2_μ): estimated as shown in section 12.4
- adjustment cost (ξ): estimated from a set of targeted moments from section 12.4[47]
- quality of information (V): estimated from set of targeted moments from section 12.4
- comovement of distortion with productivity (γ): estimated from set of targeted moments from section 12.4
- volatility of uncorrelated i.i.d. component of distortion (σ^2_ε): estimated from set of targeted moments from section 12.4
- volatility of uncorrelated permanent component of distortion (σ^2_χ): estimated from set of targeted moments from section 12.4

[44] The baseline measure uses reported book value.

[45] from NBER-CES database mentioned above; the average industry wage is calculated as total industry-wide payroll divided by total employees.

[46] before depreciation and amortization, series OIBDP in Compustat

[47] Targeted moments of variables include correlation of investment growth with lagged innovation in productivity ($\rho_{\iota,a_{-1}}$), the autocorrelation of investment growth ($\rho_{\iota,\iota_{-1}}$), the variance of investment growth (σ^2_ι), the correlation of the average product of capital with productivity ($\rho_{arpk,a}$) and overall dispersion in average product of capital (σ^2_{arpk}), the last moment of which increases with σ^2_χ.

Table 2. Parameters

Parameter	Description	Target/Value
Preferences/production		
θ	Elasticity of substitution	6
β	Discount rate	0.95
δ	Depreciation	0.10
$\hat{\alpha}_1$	Capital share	0.33 US/0.50 China
$\hat{\alpha}_2$	Labor share	0.67 US/0.50 China
Productivity/frictions		
ρ	Persistence of productivity	$\rho_{a,a_{-1}}$
σ^2_μ	Shocks to productivity	σ^2_a
$\mathbb{V}$	Signal precision	$\rho_{\iota,a_{-1}}$
ξ	Adjustment costs	$\rho_{\iota,\iota_{-1}}$
γ	Correlated factors	$\rho_{arpk,a}$
σ^2_ε	Transitory factors	σ^2_ι
σ^2_χ	Permanent factors	σ^2_{arpk}

An Aside: Empirical.
Table 2: target moments for baseline model. The file: $Table2\_US.do$ calculates the moments of U.S. firms in Table 2. It also runs the regressions described in Section 6.3 and the additional moments (inaction) for Table 6. The do file contains the following procedures, and is attached with comments.

- Use Compustat North America dataset and keep samples between 1998-2009 that are
  - incorporated in the U.S.
  - reported currency: USD
  - without multiple observations in any single year
- sort sample by firm and year, and generate variables measured according to section 12.4
  - value added & its growth
  - capital stock
  - investment & its growth
  - productivity & its growth
  - average product of capital
- generate industry-year fixed effects, extract firm idiosyncratic component from each series
- calculate gross investment as net investment + depreciation
- identify firms of inaction
- trim 3% tails of each series
- drop observations with investment rates over 100% in abs. val
- keep observations with no missing value
- estimate process on fundamentals
  - persistence of prod process
  - volatility of prod. process

  - correlation b/w inv growth and lagged prod. growth
  - correlation b/w inv growth and lagged inv. growth
  - volatility of (log of) average product of capital
  - covariance b/w apk and productivity
  - correlation b/w apk and productivity
- complementary: BKR (Bils et al., 2017) calculation (estimate the role of additive measurement error): section 6.3
  - model: $\Delta va = \phi apk + \psi i - \psi(1-\lambda)intl + FE + \varepsilon$
- save moments for Table 2 (in attached log file: Table2_US.log)

Table 4: target moments for extended model. The file: $Table4Moments\_US.do$ calculates the moments in Table 4. Its merges the Compustat data with the NBER-CES Manufacturing Industry Database in order to impute the wage bill. The do file contains the following procedures, and is attached with comments.

- Use NBER productivity data, calculate avg. wage per worker by industry within sample period 1998-2009; sort by industry and year
- Use Compustat North America dataset and keep samples between 1998-2009 that are
  - incorporated in the U.S.
  - reported currency: USD
  - without multiple observations in any single year
- merge two datasets, and drop unmatched observations
- generate variables measured according to section 12.4
  - value added & its growth
  - capital stock
  - employment
  - wage bill
  - intermediate expenditure and its share (drop those with share >1)
  - apk & apl
  - adjusted apk & apl
- take out industry-by-time fixed effects and trim 1% tails
- drop all missing data
- calculate moments for table 4
  - dispersion in adjusted apk
  - dispersion in adjusted apl
  - covariance b/w adjusted apk & apl
  - dispersion in mark-up
  - dispersion in unadjusted apk
  - dispersion of adjusted less unadjusted
- save moments for Table 4 (in attached log file: Table4_US.pdf)

12.5. * David, Schmid, and Zeke (2018, WP).

Motivation. This section provides a simple version of frictionless neoclassical investment model that motivates empirical exercises and baseline model in subsequent sections.
stylized model.

- value function of the firm:

$$V(X_t, Z_{it}, K_{it}) = \max_{K_{it}} \Pi_{it}(X_t, Z_{it}, K_{it}) - K_{it+1} + (1-\delta)K_{it} + E_t[M_{t+1}V(X_{t+1}, Z_{it+1}, K_{it+1})]$$

  where
  - X_t: aggregate fundamental shock
  - Z_{it}: idiosyncratic shock
  - Π_{it}: operational profit = revenue - labor cost
  - M_{it+1}: stochastic discount factor (correlated with aggregate shock)
  - capital law of motion:

$$K_{it+1} = (1-\delta)K_{it} + I_{it}$$

  - assumption: Cobb-Douglas technology and constant elasticity demand curves. $\Rightarrow \Pi_{it}$ is homogeneous in K_{it} of some degree $\theta < 1$ and thus

$$MPK_{it} = \theta \frac{\Pi_{it}}{K_{it}}$$

- first order condition:

$$1 = E_t[M_{t+1}(MPK_{it+1} + 1 - \delta)] \tag{393}$$

  which implies that
  - MPKs are not equalized
  - *expected discounted* MPKs are equalized
    - *expected* MPKs may not be equalized if firms load *discount* differently
    - firstly we derive an expression for *expected* MPK
- *expected* MPK:

$$\begin{aligned} 1 &= E_t[M_{t+1}(MPK_{it+1} + 1 - \delta)] \\ &= E_t[M_{t+1}]E_t[MPK_{it+1} + 1 - \delta] + cov(M_{t+1}, MPK_{it+1} + 1 - \delta) \\ &= E_t[M_{t+1}]E_t[MPK_{it+1} + 1 - \delta] + cov(M_{t+1}, MPK_{it+1}) \end{aligned} \tag{394}$$

  thus

$$\begin{aligned} &E_t[M_{t+1}]E[MPK_{it+1} + 1 - \delta] = 1 - cov(M_{t+1}, MPK_{it+1}) \\ \Rightarrow &E_t[MPK_{it+1}] = \frac{1}{E_t[M_{t+1}]} - 1 + \delta - \frac{cov(M_{t+1}, MPK_{it+1})}{E_t[M_{t+1}]} \\ \Rightarrow &E_t[MPK_{it+1}] = \underbrace{\frac{1}{E_t[M_{t+1}]} - 1 + \delta}_{\alpha_t = \text{risk free cost}} \underbrace{- \frac{cov(M_{t+1}, MPK_{it+1})}{var_t[M_{t+1}]}}_{\beta_{it} = \text{risk exposure}} \underbrace{\frac{var_t[M_{t+1}]}{E_t[M_{t+1}]}}_{\lambda_t = \text{market price of risk}} \\ \Rightarrow &E_t[MPK_{it+1}] = \alpha_t + \beta_{it}\lambda_t \end{aligned} \tag{395}$$

  which is a factor model equation that implies
  - α_t: no-arbitrage condition must equalized α_t, risk-free MPK or user cost of capital, to risk-free gross interest rate.
  - β_{it}: *expected* MPK is equalized if and only if β_{it} is equalized.

- $\sigma^2_{E_t[MPK_{it+1}]}$: cross-sectional variance of expected MPK depends on cross-sectional variance of risk exposure (conditional beta) $\sigma^2_{\beta_t}$ and price of risk:

$$\sigma^2_{E_t[MPK_{it+1}]} = \sigma^2_{\beta_t}\lambda_t^2 \tag{396}$$

empirical predictions. Several empirical predictions can be drawn and tested from last few equations.

- Exposure to SDF, β_{it}, determines both cross-sectional asset returns and *expected* MPK:
  Prediction 1: higher $\beta_{it} \Rightarrow$ higher $E_t[MPK_{it+1}]$
- Variation in price of risk, λ_t, leads to predictable variation in *mean expected* MPK:
  Prediction 2: higher $\lambda_t \Rightarrow$ higher $E[MPK_{t+1}]$
- MPK dispersion is positively related to dispersion in β:
  Prediction 3: higher $\sigma^2_{\beta_t} \Rightarrow$ higher $\sigma^2_{E_t[MPK_{it+1}]}$
- MPK dispersion is positively correlated with price of risks, λ_t:
  Prediction 4: higher $\lambda_t \Rightarrow$ higher $\sigma^2_{E_t[MPK_{it+1}]}$

Baseline Model. This section presents the concrete quantitative model of the paper that extends insight from previous sections featuring two main building blocks: stochastic discount factor and a cross-sectional heterogeneity among firms. This is a partial equilibrium model. technology.

$$Y_{it} = X_t^{\hat{\beta}_i}\hat{Z}_{it}K_{it}^{\theta_1}N_{it}^{\theta_2}, \quad \theta_1 + \theta_2 < 1 \tag{397}$$

- productivity in log: $\hat{\beta}_i x_t + \hat{z}_{it}$
- x_t: log of aggregate productivity component X_t

$$x_{t+1} = \rho_x x_t + \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim N(0, \sigma^2_\varepsilon)$$

- $\hat{\beta}_i$: captures exposure to aggregate shock and follows a normal distribution $N(\bar{\hat{\beta}}, \sigma^2_{\hat{\beta}})$
- $\hat{z}_{it}$: log of idiosyncratic productivity component $\hat{Z}_{it}$

$$\hat{z}_{it+1} = \rho_z \hat{z}_{it} + \hat{\varepsilon}_{t+1}, \quad \hat{\varepsilon}_{t+1} \sim N(0, \sigma^2_{\hat{\varepsilon}})$$

stochastic discount factor.

$$\begin{aligned} \log M_{it+1} \equiv m_{t+1} &= \log\rho - \gamma_t\varepsilon_{t+1} - \frac{1}{2}\gamma_t\sigma^2_\varepsilon \\ \gamma_t &= \gamma_0 + \gamma_1 x_t, \quad \gamma_0 > 0 \quad \gamma_1 \leq 0 \end{aligned} \tag{398}$$

One direct advantage of parameterizing SDF directly instead of modeling a consumer problem is for easing quantitative analysis with asset return. In this specification,

- SDF is determined by aggregate shock ε_{t+1}
- conditional volatility of SDF: $\sigma_m = \gamma_t\sigma_\varepsilon$
- $\gamma_1 < 0$: countercyclical price of risk
- -log ρ: constant risk free rate
- x_t is the single source of aggregate risk (i.e., macroeconomic condition)

operational profit.

$$\Pi_{it} = \max_{N_{it}} X_t^{\hat{\beta}_i} \hat{Z}_{it} K_{it}^{\theta_1} N_{it}^{\theta_2} - W_t N_{it} \tag{399}$$

static labor choice:

$$\theta_2 X_t^{\hat{\beta}_i} \hat{Z}_{it} K_{it}^{\theta_1} N_{it}^{\theta_2 - 1} = W_t$$

taken wage rate as given in the partial equilibrium

$$W_t = X_t^{\omega} \tag{400}$$

where $\omega \in [0,1]$ measures sensitivity of wage to aggregate productivity X_t. F.O.C implies:

$$\theta_2 X_t^{\hat{\beta}_i} \hat{Z}_{it} K_{it}^{\theta_1} N_{it}^{\theta_2 - 1} = X_t^{\omega} \tag{401}$$

thus operational profit can be rewritten as

$$\Pi_{it} = G X_t^{\beta_i} Z_{it} K_{it}^{\theta} \tag{402}$$

where $G = (1-\theta_2)\theta_2^{\frac{\theta_2}{1-\theta_2}}$, $\beta_i = \frac{1}{1-\theta_2}(\hat{\beta}_i - \omega\theta_2)$, $Z_{it} = \hat{Z}_{it}^{\frac{1}{1-\theta_2}}$, and $\theta \equiv \frac{\theta_1}{1-\theta_2}$.

- β_i: exposure to aggregate conditions (productivity and wage), and $\sigma_\beta^2 = (\frac{1}{1-\theta_2})^2 \sigma_{\hat{\beta}}^2$
- θ: curvature of profit function
- z_{it}: log of rescaled idiosyncratic productivity Z_{it}

$$z_{it+1} = \rho_z z_{it} + \frac{1}{1-\theta_2}\hat{\varepsilon}_{it+1} \equiv \rho_z z_{it} + \varepsilon_{it+1}, \quad \varepsilon_{it+1} \sim N(0, \sigma_{\tilde{\varepsilon}}^2)$$

investment policy rule. plugging equation (10) into equation (1) yields

$$\begin{aligned}
1 &= E_t[M_{t+1}(\theta G X_{t+1}^{\beta_i} Z_{it+1} K_{it+1}^{\theta-1} + 1 - \delta)] \\
&= \theta G E_t[M_{t+1} X_{t+1}^{\beta_i} Z_{it+1} K_{it+1}^{\theta-1}] + (1-\delta) E_t[M_{t+1}] \\
&= \theta G E_t[e^{m_{t+1} + \beta_i x_{t+1} + z_{it+1}} K_{it+1}^{\theta-1}] + (1-\delta) E_t[M_{t+1}] \\
&= \theta G K_{it+1}^{\theta-1} E_t[e^{m_{t+1} + \beta_i x_{t+1} + z_{it+1}}] + (1-\delta) E_t[M_{t+1}] \\
&= \theta G K_{it+1}^{\theta-1} E_t[e^{\log\rho - \gamma_t \varepsilon_{t+1} - \frac{1}{2}\gamma_t \sigma_\varepsilon^2 + \beta_i x_{t+1} + z_{it+1}}] + (1-\delta) E_t[e^{\log\rho - \gamma_t\varepsilon_{t+1} - \frac{1}{2}\gamma_t\sigma_\varepsilon^2}] \\
&= \theta G K_{it+1}^{\theta-1} E_t[e^{\log\rho - \gamma_t\varepsilon_{t+1} - \frac{1}{2}\gamma_t\sigma_\varepsilon^2 + \beta_i\rho_x x_t + \beta_i\varepsilon_{t+1} + \rho_z z_{it} + \varepsilon_{it+1}}] + (1-\delta) e^{\log\rho + \frac{1}{2}\gamma_t^2\sigma_\varepsilon^2 - \frac{1}{2}\gamma_t^2\sigma_\varepsilon^2} \\
&= \theta G K_{it+1}^{\theta-1} e^{\log\rho + \rho_z z_{it} + \beta_i\rho_x x_t + \frac{1}{2}\sigma_{\tilde{\varepsilon}}^2 + \frac{1}{2}\beta_i^2\sigma_\varepsilon^2 - \beta_i\gamma_t\sigma_\varepsilon^2} + (1-\delta)\rho
\end{aligned}$$

This Euler equation implies optimal capital policy rule as

$$\theta G K_{it+1}^{\theta-1} = \frac{1-(1-\delta)\rho}{e^{\log\rho + \rho_z z_{it} + \beta_i\rho_x x_t + \frac{1}{2}\sigma_{\tilde{\varepsilon}}^2 + \frac{1}{2}\beta_i^2\sigma_\varepsilon^2 - \beta_i\gamma_t\sigma_\varepsilon^2}}$$

taking log of both sides

$$\log\theta + \log G + (\theta-1)k_{it+1} = \log(1-(1-\delta)\rho) - (\log\rho + \rho_z z_{it} + \beta_i\rho_x x_t + \frac{1}{2}\sigma_{\tilde{\varepsilon}}^2 + \frac{1}{2}\beta_i^2\sigma_\varepsilon^2 - \beta_i\gamma_t\sigma_\varepsilon^2)$$

after some rearrangement

$$k_{it+1} = \frac{1}{1-\theta}[\log\theta + \log G + \log\rho - \log(1-(1-\delta)\rho) + \beta_i\rho_x x_t + \rho_z z_{it} + \frac{1}{2}\sigma_{\tilde{\varepsilon}}^2 + \frac{1}{2}\beta_i^2\sigma_\varepsilon^2 - \beta_i\gamma_t\sigma_\varepsilon^2]$$

or equivalently

$$k_{it+1} = \frac{1}{1-\theta}[\tilde{\alpha} + \beta_i\rho_x x_t + \rho_z z_{it} + \frac{1}{2}\sigma_{\tilde{\varepsilon}}^2 + \frac{1}{2}\beta_i^2\sigma_\varepsilon^2 - \beta_i\gamma_t\sigma_\varepsilon^2]$$

where

- $\tilde{\alpha} \equiv \log\theta + \log G - \alpha$
- $\alpha \equiv -\log\rho + \log(1-(1-\delta)\rho) = r_f + \log(1-(1-\delta)\rho)$

Suppressing two terms reflecting variance of shocks[48], the capital policy function is

$$k_{it+1} = \frac{1}{1-\theta}[\tilde{\alpha} + \beta_i\rho_x x_t + \rho_z z_{it} - \beta_i\gamma_t\sigma_\varepsilon^2] \tag{403}$$

- capital increases with x_t and z_{it}: aggregate and idiosyncratic productivity
- capital decreases with β_i: exposure to risks [49]

mpk dispersion.

$$MPK_{it} = \theta\frac{\Pi_{it}}{K_{it}}$$

implies that the in log term, realized MPK is given by

$$\begin{aligned} mpk_{it+1} &= \log\theta + \pi_{it+1} - k_{it+1} \\ &= \log\theta + \log G + z_{it+1} + \beta_i x_{t+1} + \theta k_{it+1} - k_{it+1} \\ &= \log\theta + \log G + z_{it+1} + \beta_i x_{t+1} - (1-\theta)\frac{1}{1-\theta}[\tilde{\alpha} + \beta_i\rho_x x_t + \rho_z z_{it} - \beta_i\gamma_t\sigma_\varepsilon^2] \\ &= \alpha + \underbrace{\varepsilon_{it+1} + \beta_i\varepsilon_{t+1}}_{\text{uncertainty}} + \underbrace{\beta_i\gamma_t\sigma_\varepsilon^2}_{\text{risk premium}} \end{aligned} \tag{404}$$

Thus *expected mpk*, i.e. persistent components of mpk, is

$$E_t[mpk_{it+1}] = \underbrace{\alpha}_{\text{user cost of capital}} + \underbrace{\beta_i\gamma_t\sigma_\varepsilon^2}_{\text{risk premium}} \tag{405}$$

cross-sectional variance is

$$\sigma^2_{E_t[mpk_{it+1}]} = \sigma^2_\beta(\gamma_t\sigma^2_\varepsilon)^2 \tag{406}$$

and *mean* cross-sectional variance is

$$E[\sigma^2_{E_t[mpk_{it+1}]}] = E[\sigma^2_\beta(\gamma_0+\gamma_1 x_t)^2(\sigma^2_\varepsilon)^2] = \sigma^2_\beta(\gamma_0^2+\gamma_1^2\sigma_x^2)(\sigma^2_\varepsilon)^2 = \sigma^2_\beta(\gamma_0^2+\frac{\gamma_1^2\sigma_\varepsilon^2}{1-\rho_x^2})(\sigma^2_\varepsilon)^2 \tag{407}$$

The last three equations confirm the key implications from previous section:

- *Prediction 1:* higher $\beta_i \Rightarrow$ higher $E_t[mpk_{it+1}]$
- *Prediction 2:* higher $\gamma_t \Rightarrow$ higher $E[mpk_{t+1}]$
- *Prediction 3:* higher $\sigma^2_{\beta_t} \Rightarrow$ higher $\sigma^2_{E_t[mpk_{it+1}]}$
- *Prediction 4:* higher $\gamma_t \Rightarrow$ higher $\sigma^2_{E_t[mpk_{it+1}]}$

aggregation. The algebra for aggregation is identical as that in David and Venkateswaran (2019). The output aggregate output can be expressed as

$$y_{t+1} = a_{t+1} + \theta_1 k_{t+1} + \theta_2 n_{t+1} \tag{408}$$

where k_{t+1} and n_{t+1} are aggregate capital stock and labor in log terms respectively, and a_{t+1} represents the level of aggregate TFP:

$$a_{t+1} = a*_{t+1} - \frac{1}{1}\frac{\theta_1(1-\theta_2)}{1-\theta_1-\theta_2}\sigma^2_{mpk,t+1} \tag{409}$$

[48]The authors argue these two terms are negligible and play no role in analysis of risk premium effect.

[49]This effect is stronger when γ_t is higher, aka at economic downturns. This effect exists when agents are risk averse.

as in David and Venkateswaran (2019), a^*_{t+1} denotes first-best level of TFP absent of frictions. The last equation implies that aggregate TFP decreases with capital misallocation, i.e. σ^2_{mpk}.

After utilizing equation (14), last equation can be rewritten as

$$a_{t+1} = a *_{t+1} - \frac{1}{1}\frac{\theta_1(1-\theta_2)}{1-\theta_1-\theta_2}\sigma^2_\beta(\gamma_t\sigma^2_\varepsilon)^2$$

conditionally expected TFP

$$E_t[a_{t+1}] = E_t[a^*_{t+1}] - \frac{1}{1}\frac{\theta_1(1-\theta_2)}{1-\theta_1-\theta_2}\underbrace{\sigma^2_\beta(\gamma_t\sigma^2_\varepsilon)^2}_{\text{risk premium}} \tag{410}$$

is negatively affected by dispersion in risk premium, and this risk premium effect is stronger in recessions as γ_t is countercyclical.

Unconditionally expected TFP is given by

$$\bar{a} \equiv E\{E_t[a_{t+1}]\} = a^* - \frac{1}{1}\frac{\theta_1(1-\theta_2)}{1-\theta_1-\theta_2}\sigma^2_\beta(\gamma_0^2 + \frac{\gamma_1^2\sigma^2_\varepsilon}{1-\rho_x^2})(\sigma^2_\varepsilon)^2 \tag{411}$$

which implies

- direct link from cross-sectional dispersion in risk premium to long-run TFP
- negative impact of aggregate volatility on long-run TFP:
  higher σ^2_x or $\sigma^2_\varepsilon \to$ lower $\bar{a}$

expected stock return and mpk. In this subsection we establish the direct link between cross-sectional beta and expected stock market return. We follow previous section and abstract from adjustment cost. The derivation in appendix C.4 is straight forward and more general.

Expected excess stock return in log term is given as

$$E[r^e_{t+1}] \equiv \log[E_t(R^e_{t+1})] = \psi\beta_i\gamma_t\sigma^2_\varepsilon \tag{412}$$

where $\psi = \frac{1/\rho+\delta-1}{1/\rho+\delta(1-\theta)-1}\frac{1-\rho}{1-\rho\rho_x+\rho\gamma_1\sigma^2_\varepsilon}$. Similar to *expected* mpk, firm's expected excess stock return depends on β_i and increases with countercyclical γ_t [50]. In other words, risk premia are countercyclical as well.

Volatility of expected excess return is given by

$$\sigma^2_{Er^e_{t+1}} \equiv \sigma^2_{\log[E_t(R^e_{t+1})]} = \psi^2\sigma^2_\beta(\gamma_t\sigma^2_\varepsilon)^2 \tag{413}$$

which confirms that similar to dispersion of mpk, dispersion in expected stock return is also increasing with dispersion in β. In addition, the dispersion in expected stock return is also countercyclical, as price of risks γ_t is countercyclical.

Combining equation (20) with equation (13) delivers one key finding of this part that *expected mpk is proportional to expected stock return.*[51]

$$E_t[mpk_{it+1}] = \alpha + \beta_i\gamma_t\sigma^2_\varepsilon = \alpha + \frac{1}{\psi}E[r^e_{t+1}] \tag{414}$$

---

[50]when $1-\rho\rho_x+\rho\gamma_1\sigma^2_\varepsilon$ holds, to be precise.

[51]Despite that this result doesn't hold in general non-linear settings, the intuition remains that both expected stock return and expected mpk are dependent on firm's beta.

estimation. Previous equations shed light on estimation of three key parameters of the paper:γ_0, γ_1, σ_β^2. Here we specify three equations, namely equation (23)-(25) that are used to estimate three parameters above.

- γ_0: By definition of Sharpe ratio of any individual firm is

$$SR_{it} = \frac{\beta_i \gamma_t \sigma_\varepsilon^2}{\sqrt{(\frac{1-\rho\rho_x+\rho\gamma_1\sigma_\varepsilon^2}{1-\rho\rho_z})^2 \sigma_{\tilde{\varepsilon}}^2 + \beta_i^2 \sigma_\varepsilon^2}}$$

  In this linearized system, a perfect diversified portfolio faces no idiosyncratic risk from $\sigma_{\tilde{\varepsilon}}^2$ thus

$$SR_{mt} = \frac{\beta_i \gamma_t \sigma_\varepsilon^2}{\sqrt{\beta_i^2 \sigma_\varepsilon^2}} = \gamma_t \sigma_\varepsilon = (\gamma_0 + \gamma_1 x_t)\sigma_\varepsilon$$

  which gives the expression for estimation of γ_0 from market Sharpe ratio:

$$ESR_m = E[SR_{mt}] = \gamma_0 \sigma_\varepsilon \tag{415}$$

- γ_1: From equation (20), expected return to a perfectly diversified portfolio (market) would be

$$Er_{mt+1} = \psi \bar{\beta} \gamma_t \sigma_\varepsilon^2 = \psi \bar{\beta} (\gamma_0 + \gamma_1 x_t) \sigma_\varepsilon^2$$

  which gives the expression for estimation of γ_1 from market expected excess return[52]:

$$Er_m = \psi \bar{\beta} \gamma_0 \sigma_\varepsilon^2$$

  where $\psi = \frac{1/\rho+\delta-1}{1/\rho+\delta(1-\theta)-1} \frac{1-\rho}{1-\rho\rho_x+\rho\gamma_1\sigma_\varepsilon^2}$.
- σ_β^2: From equation (21), expected dispersion in excess return is

$$E[\sigma^2_{Er^e_{t+1}}] = E[\psi^2 \sigma_\beta^2 (\gamma_t \sigma_\varepsilon^2)^2] = \psi^2 \sigma_\beta^2 \sigma_\varepsilon^4 E[(\gamma_0 + \gamma_1 x_t)^2] \tag{416}$$

  which gives the expression for estimation of σ_β^2 from expected dispersion in excess return.

Empirical Strategy and Results. In this section we revisit four empirical predictions from motivating model (in section 12.5) and baseline model (in section 12.5). Here we restate these four predictions and demonstrate the empirical strategy to test them respectively.
data, sample and measurement.

- primary data sources come from CRSP and Compustat, sample period from 1965 to 2015
  - industrial (non-financial) public firms with common equities listed on the NYSE, NASDAQ or AMEX;
  - exclude firms with missing SIC codes or coded as non-classifiable
- time series data are collected between 1973-2015, all available at authors' websites.
  - market factors data: Fama and French (1992)
  - aggregate dividend and stock market values Shiller (2005)
  - credit spread: Gilchrist and Zakrajšek (2012) or excess bond (EB) premium

We first summarize the measurement of variables before presenting the empirical results.

- capital stock, K_{it}: measured by undepreciated value of series PPENT (Compustat)
- firm's revenue, Y_{it}: measured by series SALES (Compustat)
- marginal product of capital (in log term): $mpk_{it} = y_{it} - k_{it}$.

[52]In the practice of estimation, as expected returns are not directly available, the paper chooses an asset pricing model of Fama-French.

- market capitalization: measured as price times shares outstanding (CRSP).
- profitability: measured as ratio of series EBITDA (Compustat) to series AT (Compustat)
- book debt: measured by series LCT (Compustat) + 1/2 DLTT (Compustat)
- market leverage: measured as ratio of book debt to the sum of market capitalization plus book debt
- book equity: measured as sum of series SEQ, TXDITC and PSTKL (Compustat)
- book-to-market: measured as ratio of book equity to market capitalization of the firm

empirical strategy and results.

- *Prediction 1: Exposure to SDF, β_{it}, determines both cross-sectional asset returns and expected MPK: higher $\beta_{it} \Rightarrow$ higher $E_t[MPK_{it+1}]$.*

  strategy. Non-industry adjusted portfolio sort:
  - portfolio sort: sort firms into 5 portfolios based on year t MPK (1 being the lowest)
  - rebalance the portfolios annually
  - compute the contemporaneous (r_t^e) and one-period ahead (r_{t+1}^e) equal-weighted excess stock return
  - compute excess return on a high-minus-low MPK portfolio

  Industry adjusted portfolio sort:
  - demean firm-level mpk by industry-year fixed effect
  - portfolio sort: sort firms into 5 portfolios based on year t *demeaned* MPK (1 being the lowest)
  - rebalance the portfolios annually
  - compute the contemporaneous (r_t^e) and one-period ahead (r_{t+1}^e) equal-weighted excess stock return
  - compute excess return on a high-minus-low *demeaned* MPK portfolio

  results. Non-industry adjusted portfolio sort:
  - strong and positive contemporaneous correlation between MPK and stock returns
  - excess return on MPK-HML portfolio over 8% annually
  - strong and positive correlation between MPK and one-period ahead stock returns
  - predictable excess return on MPK-HML portfolio almost 5 % annually

  Industry adjusted portfolio sort:
  - strong and positive contemporaneous correlation between MPK and stock returns
  - excess return on MPK-HML portfolio over 8% annually
  - strong and positive correlation between MPK and one-period ahead stock returns
  - predictable excess return on MPK-HML portfolio around 2.6 % annually
- *Prediction 2: Variation in price of risk, λ_t, leads to predictable variation in mean expected MPK: higher $\lambda_t \Rightarrow$ higher $E[MPK_{t+1}]$*

  strategy. Estimate a time-series regression

$$E[mpk_{it+1}] = \psi_0 + \psi_1 \lambda_t + \zeta_{t+1} \tag{417}$$

using three different proxies of the price of risks: PD ratio of aggregate stock market, GZ credit spread and EB premium.

results. Time variance in price of risks forecast future level of MPK
- coefficient on PD ratio (negatively correlated with price of risk) is negative
- coefficient on GZ spread (positively correlated with price of risk) is positive
- coefficient on EB premium (positively correlated with price of risk) is positive

- *Prediction 3: MPK dispersion is positively related to dispersion in β: higher $\sigma^2_{\beta_t} \Rightarrow$ higher $\sigma^2_{E_t[MPK_{it+1}]}$*

strategy. Estimate regressions of industry-level MPK dispersion on the dispersion in expected returns and betas

$$\sigma(mpk_{jt+1}) = \psi_0 + \psi_1 \sigma(x_{jt}) + \zeta_{jt+1}, \quad x_{jt} = E[r_{jt}] \quad or \quad \beta_{jt} \tag{418}$$

where we use lagged independent variables to avoid simultaneity bias.

results.
- industry with higher dispersion in expected stock returns and (different measures of) beta exhibit greater dispersion in MPK

- *Prediction 4: MPK dispersion is positively correlated with price of risks, λ_t: higher $\lambda_t \Rightarrow$ higher $\sigma^2_{E_t[MPK_{it+1}]}$*

strategy. Estimate a time-series regression

$$y_{t+1} = \psi_0 + \psi_1 \lambda_t + \zeta_{t+1} \quad y_{t+1} = \sigma(mpk_{t+1}) \quad or \quad r_{HML,t+1} \tag{419}$$

where $\sigma(mpk_{t+1})$ denotes within-industry standard deviation of MPK and $r_{HML,t+1}$ denotes cumulative twelve month return on the MPK-HML portfolio.

results.
- MPK dispersion is positively correlated with price of risks
- expected return on MPK-HML portfolio is positively correlated with price of risks

Extended Model. In this section we a variant of baseline model augmented with two types of capital, tangible and intangible one. We model intangible capital following Eisfeldt and Papanikolaou (2013), Li et al. (2014), Peters and Taylor (2017) etc. that firm's production requires both tangible and intangible capital, the latter often referred to as human capital, R&D, corporate governance etc, and non-capital inputs.
technology.

$$Y_{it} = X_t^{\hat{\beta}_i} \hat{Z}_{it} [(K^m_{it})^\alpha (K^u_{it})^{1-\alpha}]^{\theta_1} N_{it}^{\theta_2}, \quad \theta_1 + \theta_2 < 1 \tag{420}$$

- tangible capital: K^m_{it}
- intangible capital: K^u_{it}
- compound capital stock: $K_{it} \equiv (K^m_{it})^\alpha (K^u_{it})^{1-\alpha}$
- x_t: log of aggregate productivity component X_t

$$x_{t+1} = \rho_x x_t + \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim N(0, \sigma^2_\varepsilon)$$

- $\hat{\beta}_i$: captures exposure to aggregate shock and follows a normal distribution $N(\bar{\hat{\beta}}, \sigma^2_{\hat{\beta}})$
- $\hat{z}_{it}$: log of idiosyncratic productivity component $\hat{Z}_{it}$

$$\hat{z}_{it+1} = \rho_z \hat{z}_{it} + \hat{\varepsilon}_{t+1}, \quad \hat{\varepsilon}_{t+1} \sim N(0, \sigma^2_{\hat{\varepsilon}})$$

stochastic discount factor.

$$\begin{aligned} \log M_{it+1} \equiv m_{t+1} = \log\rho - \gamma_t \varepsilon_{t+1} - \frac{1}{2}\gamma_t \sigma_\varepsilon^2 \\ \gamma_t = \gamma_0 + \gamma_1 x_t, \quad \gamma_0 > 0 \quad \gamma_1 \leq 0 \end{aligned} \tag{421}$$

One direct advantage of parameterizing SDF directly instead of modeling a consumer problem is for easing quantitative analysis with asset return. In this specification,

- SDF is determined by aggregate shock ε_{t+1}
- conditional volatility of SDF: $\sigma_m = \gamma_t \sigma_\varepsilon$
- $\gamma_1 < 0$: countercyclical price of risk
- -$\log\rho$: constant risk free rate
- x_t is the single source of aggregate risk (i.e., macroeconomic condition)

operational profit.

$$\Pi_{it} = \max_{N_{it}} X_t^{\hat{\beta}_i} \hat{Z}_{it} K_{it}^{\theta_1} N_{it}^{\theta_2} - W_t N_{it} \tag{422}$$

static labor choice:

$$\theta_2 X_t^{\hat{\beta}_i} \hat{Z}_{it} K_{it}^{\theta_1} N_{it}^{\theta_2 - 1} = W_t$$

taken wage rate as given in the partial equilibrium

$$W_t = X_t^\omega \tag{423}$$

where $\omega \in [0,1]$ measures sensitivity of wage to aggregate productivity X_t. F.O.C implies:

$$\theta_2 X_t^{\hat{\beta}_i} \hat{Z}_{it} K_{it}^{\theta_1} N_{it}^{\theta_2 - 1} = X_t^\omega \tag{424}$$

thus operational profit can be rewritten as

$$\Pi_{it} = G X_t^{\beta_i} Z_{it} K_{it}^\theta = G X_t^{\beta_i} Z_{it} [(K_{it}^m)^\alpha (K_{it}^u)^{1-\alpha}]^\theta \tag{425}$$

where $G = (1-\theta_2)\theta_2^{\frac{\theta_2}{1-\theta_2}}$, $\beta_i = \frac{1}{1-\theta_2}(\hat{\beta}_i - \omega\theta_2)$, $Z_{it} = \hat{Z}_{it}^{\frac{1}{1-\theta_2}}$, and $\theta \equiv \frac{\theta_1}{1-\theta_2}$.

- β_i: exposure to aggregate conditions (productivity and wage), and $\sigma_\beta^2 = (\frac{1}{1-\theta_2})^2 \sigma_{\hat{\beta}}^2$
- θ: curvature of profit function
- z_{it}: log of rescaled idiosyncratic productivity Z_{it}

$$z_{it+1} = \rho_z z_{it} + \frac{1}{1-\theta_2}\hat{\varepsilon}_{it+1} \equiv \rho_z z_{it} + \varepsilon_{it+1}, \quad \varepsilon_{it+1} \sim N(0, \sigma_{\tilde{\varepsilon}}^2)$$

investment policy rule.

- Investment rule for physical capital is:

$$\begin{aligned}
1 &= E_t[M_{t+1}(\theta G X_{t+1}^{\beta_i} Z_{it+1} K_{it+1}^{\theta-1} \alpha \frac{K_{it+1}}{K_{it+1}^m} + 1 - \delta^m)] \\
&= \alpha\theta G E_t[M_{t+1} X_{t+1}^{\beta_i} Z_{it+1} K_{it+1}^{\theta} (K_{it+1}^m)^{-1}] + (1-\delta^m) E_t[M_{t+1}] \\
&= \alpha\theta G E_t[e^{m_{t+1}+\beta_i x_{t+1}+z_{it+1}} K_{it+1}^{\theta} (K_{it+1}^m)^{-1}] + (1-\delta^m) E_t[M_{t+1}] \\
&= \alpha\theta G K_{it+1}^{\theta} (K_{it+1}^m)^{-1} E_t[e^{m_{t+1}+\beta_i x_{t+1}+z_{it+1}}] + (1-\delta^m) E_t[M_{t+1}] \\
&= \alpha\theta G K_{it+1}^{\theta} (K_{it+1}^m)^{-1} E_t[e^{\log\rho-\gamma_t\varepsilon_{t+1}-\frac{1}{2}\gamma_t\sigma_\varepsilon^2+\beta_i x_{t+1}+z_{it+1}}] + (1-\delta^m) E_t[e^{\log\rho-\gamma_t\varepsilon_{t+1}-\frac{1}{2}\gamma_t\sigma_\varepsilon^2}] \\
&= \alpha\theta G K_{it+1}^{\theta} (K_{it+1}^m)^{-1} E_t[e^{\log\rho-\gamma_t\varepsilon_{t+1}-\frac{1}{2}\gamma_t\sigma_\varepsilon^2+\beta_i\rho_x x_t+\beta_i\varepsilon_{t+1}+\rho_z z_{it}+\varepsilon_{it+1}}] + (1-\delta^m) e^{\log\rho+\frac{1}{2}\gamma_t^2\sigma_\varepsilon^2-\frac{1}{2}\gamma_t^2\sigma_\varepsilon^2} \\
&= \alpha\theta G K_{it+1}^{\theta} (K_{it+1}^m)^{-1} e^{\log\rho+\rho_z z_{it}+\beta_i\rho_x x_t+\frac{1}{2}\sigma_{\tilde\varepsilon}^2+\frac{1}{2}\beta_i^2\sigma_\varepsilon^2-\beta_i\gamma_t\sigma_\varepsilon^2} + (1-\delta^m)\rho \\
&= \alpha\theta G (K_{it+1}^m)^{\alpha\theta-1} (K_{it+1}^u)^{(1-\alpha)\theta} e^{\log\rho+\rho_z z_{it}+\beta_i\rho_x x_t+\frac{1}{2}\sigma_{\tilde\varepsilon}^2+\frac{1}{2}\beta_i^2\sigma_\varepsilon^2-\beta_i\gamma_t\sigma_\varepsilon^2} + (1-\delta^m)\rho
\end{aligned}$$

- Investment rule for intangible capital is:

$$\begin{aligned}
1 &= E_t[M_{t+1}(\theta G X_{t+1}^{\beta_i} Z_{it+1} K_{it+1}^{\theta-1} \alpha \frac{K_{it+1}}{K_{it+1}^m} + 1 - \delta^u)] \\
&= (1-\alpha)\theta G K_{it+1}^{\theta} (K_{it+1}^u)^{-1} e^{\log\rho+\rho_z z_{it}+\beta_i\rho_x x_t+\frac{1}{2}\sigma_{\tilde\varepsilon}^2+\frac{1}{2}\beta_i^2\sigma_\varepsilon^2-\beta_i\gamma_t\sigma_\varepsilon^2} + (1-\delta^u)\rho \\
&= (1-\alpha)\theta G (K_{it+1}^m)^{\alpha\theta} (K_{it+1}^u)^{(1-\alpha)\theta-1} e^{\log\rho+\rho_z z_{it}+\beta_i\rho_x x_t+\frac{1}{2}\sigma_{\tilde\varepsilon}^2+\frac{1}{2}\beta_i^2\sigma_\varepsilon^2-\beta_i\gamma_t\sigma_\varepsilon^2} + (1-\delta^m)\rho
\end{aligned}$$

- Two Euler equations imply optimal capital policy rules are

$$\alpha\theta G (K_{it+1}^m)^{\alpha\theta-1} (K_{it+1}^u)^{(1-\alpha)\theta} = \frac{1-(1-\delta^m)\rho}{e^{\log\rho+\rho_z z_{it}+\beta_i\rho_x x_t+\frac{1}{2}\sigma_{\tilde\varepsilon}^2+\frac{1}{2}\beta_i^2\sigma_\varepsilon^2-\beta_i\gamma_t\sigma_\varepsilon^2}}$$

  for physical capital and

$$(1-\alpha)\theta G (K_{it+1}^m)^{\alpha\theta} (K_{it+1}^u)^{(1-\alpha)\theta-1} = \frac{1-(1-\delta^u)\rho}{e^{\log\rho+\rho_z z_{it}+\beta_i\rho_x x_t+\frac{1}{2}\sigma_{\tilde\varepsilon}^2+\frac{1}{2}\beta_i^2\sigma_\varepsilon^2-\beta_i\gamma_t\sigma_\varepsilon^2}}$$

  for intangible capital respectively.
- taking log of both sides yields

$$\begin{aligned}
&\log\alpha + \log\theta + \log G + (\alpha\theta-1)k_{it+1}^m + (1-\alpha)\theta k_{it+1}^u \\
&= \log[1-(1-\delta^m)\rho] - (\log\rho + \rho_z z_{it} + \beta_i\rho_x x_t + \frac{1}{2}\sigma_{\tilde\varepsilon}^2 + \frac{1}{2}\beta_i^2\sigma_\varepsilon^2 - \beta_i\gamma_t\sigma_\varepsilon^2)
\end{aligned}$$

  and

$$\begin{aligned}
&\log(1-\alpha) + \log\theta + \log G + \alpha\theta k_{it+1}^m + (\theta-\alpha\theta-1)k_{it+1}^u \\
&= \log[1-(1-\delta^u)\rho] - (\log\rho + \rho_z z_{it} + \beta_i\rho_x x_t + \frac{1}{2}\sigma_{\tilde\varepsilon}^2 + \frac{1}{2}\beta_i^2\sigma_\varepsilon^2 - \beta_i\gamma_t\sigma_\varepsilon^2)
\end{aligned}$$

- after some rearrangement

$$\begin{aligned}
k_{it+1}^m = \frac{1}{1-\theta}\{&\log\theta + \log G + (\log\rho + \beta_i\rho_x x_t + \rho_z z_{it} + \frac{1}{2}\sigma_{\tilde\varepsilon}^2 + \frac{1}{2}\beta_i^2\sigma_\varepsilon^2 - \beta_i\gamma_t\sigma_\varepsilon^2) \\
&+ \log\alpha - \log[1-(1-\delta^m)\rho] + (1-\alpha)\theta\log[\frac{1-\alpha}{\alpha}\frac{1-(1-\delta^m)\rho}{1-(1-\delta^u)\rho}]\}
\end{aligned}$$

and

$$k^u_{it+1} = \frac{1}{1-\theta}\{\log\theta + \log G + (\log\rho + \beta_i\rho_x x_t + \rho_z z_{it} + \frac{1}{2}\sigma^2_{\tilde{\varepsilon}} + \frac{1}{2}\beta_i^2\sigma^2_{\varepsilon} - \beta_i\gamma_t\sigma^2_{\varepsilon}) + \log(1-\alpha) - \log[1-(1-\delta^u)\rho] + \alpha\theta\log[\frac{\alpha}{1-\alpha}\frac{1-(1-\delta^u)\rho}{1-(1-\delta^m)\rho}]\}$$

- or equivalently

$$k^m_{it+1} = \frac{1}{1-\theta}\{\tilde{\alpha}^m + \beta_i\rho_x x_t + \rho_z z_{it} + \frac{1}{2}\sigma^2_{\tilde{\varepsilon}} + \frac{1}{2}\beta_i^2\sigma^2_{\varepsilon} - \beta_i\gamma_t\sigma^2_{\varepsilon}\}$$

and

$$k^u_{it+1} = \frac{1}{1-\theta}\{\tilde{\alpha}^u + \beta_i\rho_x x_t + \rho_z z_{it} + \frac{1}{2}\sigma^2_{\tilde{\varepsilon}} + \frac{1}{2}\beta_i^2\sigma^2_{\varepsilon} - \beta_i\gamma_t\sigma^2_{\varepsilon}\}$$

. where
  - $\tilde{\alpha}^m \equiv \log\theta + \log G + \log\rho - \log[1-(1-\delta^m)\rho] + \log\alpha + (1-\alpha)\theta\log[\frac{1-\alpha}{\alpha}\frac{1-(1-\delta^m)\rho}{1-(1-\delta^u)\rho}]$
    $\equiv \log\theta + \log G - \alpha^m$
  - $\tilde{\alpha}^u \equiv \log\theta + \log G + \log\rho - \log[1-(1-\delta^u)\rho] + \log(1-\alpha) + \alpha\theta\log[\frac{\alpha}{1-\alpha}\frac{1-(1-\delta^u)\rho}{1-(1-\delta^m)\rho}]$
    $\equiv \log\theta + \log G - \alpha^u$
  - ($\tilde{\alpha} \equiv \log\theta + \log G + \log\rho - \log[1-(1-\delta)\rho)]$ in baseline model)
- *Suppressing two terms reflecting variance of shocks*[53], the capital policy function is

$$k^j_{it+1} = \frac{1}{1-\theta}[\tilde{\alpha}^j + \beta_i\rho_x x_t + \rho_z z_{it} - \beta_i\gamma_t\sigma^2_{\varepsilon}] \quad j = m, n \tag{426}$$

  - *physical/intangible* capital increases with x_t and z_{it}: aggregate and idiosyncratic productivity
  - *physical/intangible* capital decreases with β_i: exposure to risks [54]

mpk dispersion.

- *total mpk*: define total mpk as $\partial\Pi_{it}/\partial K_{it}$:

$$MPK_{it+1} = \theta\frac{\Pi_{it+1}}{K_{it+1}}$$

implies that the in log term, realized MPK is given by

$$\begin{aligned}
mpk_{it+1} &= \log\theta + \pi_{it+1} - k_{it+1} \\
&= \log\theta + \log G + z_{it+1} + \beta_i x_{t+1} + \theta k_{it+1} - k_{it+1} \\
&= \log\theta + \log G + z_{it+1} + \beta_i x_{t+1} - (1-\theta)[\alpha k^m_{it+1} + (1-\alpha)k^u_{it+1}] \\
&= \log\theta + \log G + z_{it+1} + \beta_i x_{t+1} - (1-\theta)\alpha k^m_{it+1} - (1-\theta)(1-\alpha)k^u_{it+1} \\
&= \log\theta + \log G + z_{it+1} + \beta_i x_{t+1} \\
&- \alpha\{\tilde{\alpha}^m + \beta_i\rho_x x_t + \rho_z z_{it} - \beta_i\gamma_t\sigma^2_{\varepsilon}\} - (1-\alpha)\{\tilde{\alpha}^u + \beta_i\rho_x x_t + \rho_z z_{it} - \beta_i\gamma_t\sigma^2_{\varepsilon}\} \\
&= \log\theta + \log G + z_{it+1} + \beta_i x_{t+1} - \beta_i\rho_x x_t - \rho_z z_{it} - \beta_i\gamma_t\sigma^2_{\varepsilon} - [\alpha\tilde{\alpha}^m + (1-\alpha)\tilde{\alpha}^u] \\
&= \underbrace{\alpha\alpha^m + (1-\alpha)\alpha^u}_{\text{user cost of compound capital}} + \underbrace{\varepsilon_{it+1} + \beta_i\varepsilon_{t+1}}_{\text{uncertainty}} + \underbrace{\beta_i\gamma_t\sigma^2_{\varepsilon}}_{\text{risk premium}}
\end{aligned}$$

[53]The authors argue these two terms are negligible and play no role in analysis of risk premium effect.

[54]This effect is stronger when γ_t is higher, aka at economic downturns. This effect exists when agents are risk averse.

- *expected mpk*, i.e. persistent components of mpk, is

$$E_t[mpk_{it+1}] = \alpha\alpha^m + (1-\alpha)\alpha^u + \underbrace{\beta_i\gamma_t\sigma_\varepsilon^2}_{\text{risk premium}} \tag{427}$$

- *cross-sectional variance* is

$$\sigma^2_{E_t[mpk_{it+1}]} = \sigma^2_\beta(\gamma_t\sigma^2_\varepsilon)^2 \tag{428}$$

- *mean* cross-sectional variance is

$$E[\sigma^2_{E_t[mpk_{it+1}]}] = E[\sigma^2_\beta(\gamma_0+\gamma_1 x_t)^2(\sigma^2_\varepsilon)^2] = \sigma^2_\beta(\gamma_0^2+\gamma_1^2\sigma_x^2)(\sigma^2_\varepsilon)^2 = \sigma^2_\beta(\gamma_0^2+\frac{\gamma_1^2\sigma_\varepsilon^2}{1-\rho_x^2})(\sigma^2_\varepsilon)^2 \tag{429}$$

- The last three equations confirm that key implications from baseline model hold for our extended model where capital is compounded of physical and intangible type.
  - *Prediction 1:* higher $\beta_i \Rightarrow$ higher $E_t[mpk_{it+1}]$
  - *Prediction 2:* higher $\gamma_t \Rightarrow$ higher $E[mpk_{t+1}]$
  - *Prediction 3:* higher $\sigma^2_{\beta_t} \Rightarrow$ higher $\sigma^2_{E_t[mpk_{it+1}]}$
  - *Prediction 4:* higher $\gamma_t \Rightarrow$ higher $\sigma^2_{E_t[mpk_{it+1}]}$

physical mpk dispersion.

- *physical mpk*: define physical mpk as $\partial\Pi_{it}/\partial K^m_{it}$:

$$MPK^m_{it+1} = \alpha\theta G X^{\beta_i}_{t+1} Z_{it+1}(K^m_{it+1})^{\alpha\theta-1}(K^u_{it+1})^{(1-\alpha)\theta}$$

implies that the in log term, realized MPK is given by

$$\begin{aligned}
mpk^m_{it+1} &= \log\alpha+\log\theta+\log G+\beta_i x_{t+1}+z_{it+1}+(\alpha\theta-1)k^m_{it+1}+(1-\alpha)\theta k^u_{it+1}\\
&= \log\alpha+\log\theta+\log G+\beta_i x_{t+1}+z_{it+1}+\frac{1}{1-\theta}\{(\alpha\theta-1)[\tilde\alpha^m+\beta_i\rho_x x_t+\rho_z z_{it}-\beta_i\gamma_t\sigma^2_\varepsilon]\\
&+(\theta-\alpha\theta)[\tilde\alpha^u+\beta_i\rho_x x_t+\rho_z z_{it}-\beta_i\gamma_t\sigma^2_\varepsilon]\}\\
&= \log\alpha+\log\theta+\log G+\beta_i\varepsilon_{t+1}+\varepsilon_{it+1}+\beta_i\gamma_t\sigma^2_\varepsilon+\frac{1}{1-\theta}\{(\alpha\theta-1)\tilde\alpha^m+(\theta-\alpha\theta)\tilde\alpha^u\}\\
&= \underbrace{\frac{1-\alpha\theta}{1-\theta}(\log\alpha+\alpha^m)-\frac{(1-\alpha)\theta}{1-\theta}(\log\alpha+\alpha^u)}_{\text{user cost of physical capital}}+\underbrace{\varepsilon_{it+1}+\beta_i\varepsilon_{t+1}}_{\text{uncertainty}}+\underbrace{\beta_i\gamma_t\sigma^2_\varepsilon}_{\text{risk premium}}
\end{aligned}$$

- *expected mpk*, i.e. persistent components of mpk, is

$$E_t[mpk^m_{it+1}] = \frac{1-\alpha\theta}{1-\theta}(\log\alpha+\alpha^m)-\frac{(1-\alpha)\theta}{1-\theta}(\log\alpha+\alpha^u)+\underbrace{\beta_i\gamma_t\sigma^2_\varepsilon}_{\text{risk premium}} \tag{430}$$

- *cross-sectional variance* is

$$\sigma^2_{E_t[mpk^m_{it+1}]} = \sigma^2_\beta(\gamma_t\sigma^2_\varepsilon)^2 = \sigma^2_{E_t[mpk_{it+1}]} \tag{431}$$

- *mean* cross-sectional variance is

$$E[\sigma^2_{E_t[mpk^m_{it+1}]}] = E[\sigma^2_\beta(\gamma_0+\gamma_1 x_t)^2(\sigma^2_\varepsilon)^2] = \sigma^2_\beta(\gamma_0^2+\frac{\gamma_1^2\sigma_\varepsilon^2}{1-\rho_x^2})(\sigma^2_\varepsilon)^2 = E[\sigma^2_{E_t[mpk_{it+1}]}] \tag{432}$$

- The last three equations confirm that key implications from baseline model hold for our extended model where intangible capital is introduced:
  - *Prediction 1:* higher $\beta_i \Rightarrow$ higher $E_t[mpk^m_{it+1}]$
  - *Prediction 2:* higher $\gamma_t \Rightarrow$ higher $E[mpk^m_{t+1}]$

- *Prediction 3:* higher $\sigma^2_{\beta_t}$ $\Rightarrow$ higher $\sigma^2_{E_t[mpk^m_{it+1}]}$
- *Prediction 4:* higher γ_t $\Rightarrow$ higher $\sigma^2_{E_t[mpk^m_{it+1}]}$

13. Belief Heterogeneity

13.1. *** Geanakoplos (2010).** This paper studies models of the leverage cycle arising from belief heterogeneity among investors. The paper shows, as a result, market equilibrium leverage becomes too high in boom and too low in recession, amplifying fluctuations through the mechanism: 'scary bad news' → increases uncertainty & volatility of asset returns → lower leverage → lower prices → redistribution of wealth from optimists to pessimists → lower leverage & prices

The two-period model shows how that the price of an asset rises when it can be leveraged more, and the equilibrium promise ensures no default. The three-period model carries the insight of no-default leverage principle from two-period model to further shows a maturity mismatch problem and the aforementioned mechanism.

Two-period Model.
Set-up. Discrete, two period t= 0,1, with two states in last period: Up(U) or Down(D). There is a risky asset, with

- no payoff in period 0;
- final payoff: depends on state (up: pay 1; down: pay 0.2.)
- risky-free interest rate is zero

Investors are risk-neutral with heterogeneous belief, indexed by h $\in (0,1)$

- each endowed with 1 unit of consumption good and 1 unit of asset
- agents can trade their endowed asset at period 0
- h thinks probability of Up is h.
- h follows a uniform distribution over $(0,1)$

No Borrowing. Denote the price of asset at period 0 as p. It's straightforward that agents with

$$h * 1 + (1-h) * 0.2 > p \quad or \quad h > \frac{p-0.2}{0.8} \equiv \hat{h}$$

will buy as much as possible and others will sell endowed asset. The market clearing condition, $(1-\hat{h}) * 1 = \hat{h}p$

$$1 - \frac{p-0.2}{0.8} = \frac{p-0.2}{0.8}p$$

gives equilibrium price of $p = 2/3$ and $\hat{h} = 0.6$.
Exogenous Leverage. We now assume non-contingent borrowing contract, with promises of the same amount φ in both states. Thus repayment under two states are

$$\min\{\varphi, 1\} \quad \text{if state is Up}$$

$$\min\{\varphi, 0.2\} \quad \text{if state is Down}$$

Note the previous case of no borrowing is a special condition with $\varphi = 0$. Now consider a natural limit by setting $\varphi = 0.2$, such that no default is to occur in either case. Now denote the marginal buyer as $\tilde{h}$:

$$\underbrace{(1-0.2)\tilde{h} + 0 * (1-\tilde{h})}_{\text{expected return}} = \underbrace{p - 0.2}_{\text{down payment}}, \quad \Leftrightarrow \tilde{h} = \frac{p-0.2}{0.8},$$

and market clearing condition that $(1-\tilde{h}) * 1 + \varphi = \tilde{h}p$

$$p = \frac{(1-\tilde{h}) * 1 + 0.2}{\tilde{h}}$$

gives equilibrium price of $p = 0.75$ and $\tilde{h} = 0.69$.
Equilibrium leverage ratio is

$$leverage = \frac{p}{p - \varphi} = 1.4,$$

which corresponds to loan-to-value ratio of 0.27.
Endogenous Leverage. We revisit the problem with borrowing, but now allow agent to choose a loan contract, each characterized by a pair of (promise, collateral). Given property of homogeneity of degree one, we restrict our attention on contract backed by collateral of one unit of asset. Given the state-dependent (actual) repayment, each unit of loan contract can be traded at price of π_j, for example, one unit of loan contract without default risk ($\varphi_j \leq 0.2$) is simply priced at one over risk-free rate.

The equilibrium contract being traded at the market is one with promise 0.2. In other word, the allocation coincides with example of exogenous leverage we discussed, with marginal buyer being $\tilde{h} = 0.69$ and $p = 0.075$. We can also solve the interest rate (price) of other non-traded loan contract:

- if $\varphi_j \leq 0.2$, $\pi_j = 1/(1 + j) = 1$
- if $\varphi_j \in (0.2, 1)$, $\pi_j = \tilde{h}\varphi_j + (1 - \tilde{h})0.2$
- if $\varphi_j \geq 1$, $\pi_j = \tilde{h}1 + (1 - \tilde{h})0.2$

It's also straightforward why only the contract $\varphi_j = 0.2$ is chosen. Let's consider the cost and benefit of borrowing more. By choosing a higher leverage the borrowers can get more funding at the beginning, but at a cost of repaying more (in good state). For example, by increasing φ_j from 0.2 to 0.3, the borrowers h_j get $0.1\tilde{h}$ more at the beginning, but have to repay $0.1h_j$ more in his expectation. Thus, the equilibrium contract being chosen is the one involving zero default risk.

Three-period Model.
Set-up. Discrete, three period t= 0,1,2, with two states in period 1 and 2: Up(U) or Down(D). There is a risky asset, with

- no payoff in period 0 and 1
- possible realization at period 2: UU, UD, DU, DD
- final payoff: depends on state (DD: pay 0.2; otherwise 1 [55])
- risky-free interest rate is zero

Investors are risk-neutral with heterogeneous belief, indexed by h $\in (0, 1)$

- each endowed with 1 unit of consumption good and 1 unit of asset
- agents can trade their endowed asset at period 0 and 1
- h thinks probability of Up is h, i.i.d. across states
- h follows a uniform distribution over $(0, 1)$

Loan Contract. We assume loan contracts are one-period loans. Inheriting the insight from two-period model, the equilibrium contract bears no default in the margin at each period/state:

- at period 0, $\varphi_0 = p_D$
- if realized state is D, $\varphi_D = 0.2$
- if realized state is U, there is no uncertainty $\rightarrow$ $\varphi_U = 1$ & $p_U = 1$ $\rightarrow$ agent with $h = \bar{h} = 1$ hold all asset (first-best)

[55]We will generalize the assumption later.

Now we solve the allocation in period 0 and period 1: state D.
Equilibrium. Let's denote $\hat{h_0}$ and $\hat{h_D}$ to be the marginal buyer at period 0 and state D at period 1. Note that if realized state is D at period 1, original buyers default and liquidate all asset they hold. The following equilibrium conditions can be used to solve $\{p_0, \hat{h_0}, p_D, \hat{h_D}\}$:

- Euler equation: The marginal buyer in state D at period 1 must be indifferent b/w buying or not:

$$\hat{h_D} * (1 - \varphi_D) + (1 - \hat{h_D}) * 0 = p_D - \varphi_D$$

- market clearing condition in state D at period 1:

$$(\hat{h_0} - \hat{h_D}) * \frac{1}{\hat{h_0}} + \varphi_D = \frac{\hat{h_D}}{\hat{h_0}} p_D \quad \rightarrow \quad p_D = \frac{\hat{h_0}(1 + \varphi_D) - \hat{h_D}}{\hat{h_D}}$$

- market clearing condition at period 0:

$$(1 - \hat{h_0}) * 1 + \varphi_0 = \hat{h_0} p_0 \quad \rightarrow \quad p_0 = \frac{(1 - \hat{h_0}) * 1 + p_D}{\hat{h_0}}$$

- Euler equation: The marginal buyer at period 0 must be indifferent b/w buying or not:

$$\frac{\hat{h_0} * (1 - \varphi_0) + (1 - \hat{h_0}) * 0}{p_0 - \varphi_0} = \hat{h_0} * 1 + (1 - \hat{h_0}) \frac{\hat{h_0}(1 - \varphi_D)}{p_D - \varphi_D}$$

As a result, at period 0, $\hat{h_0} = 0.87$ and $p_0 = 0.95$; in state D at period 1, $\hat{h_D} = 0.61$ and $p_D = 0.69$. The implied equilibrium leverage in two states (periods) are 3.6 and 1.4.
Crash in Price and Leverage. There are three forces accounting for the crash in state D at period 1.

- fundamental: the realization of bad news.
- loss of natural buyers: leveraged buyers at period 0 go bankrupt
- deleveraging process: the margin increases from 0.27 to 0.71; the leverage decreases from 3.6 to 1.4.

Discussion 1: Cash Holding. Given the nature that price (return) is much lower (higher) than usual, some optimists may choose to hold cash at initial period. For example, the agent $h = \hat{h_0} - \varepsilon$ believes one unit of asset gives $0.13^2 * 0.2 + (1 - 0.13)^2 * 1 > 0.98$. he is still not willing to buy it at period 0.

13.2. *** Martin (2020).** This paper studies the effect of belief heterogeneity on asset price. The paper shows that the asset market is over-represented (in terms of wealth) by optimistic investors in good state and by pessimistic investor in bad state, thus market sentiment emerges endogenously to affect asset price, and to induce speculations.

The paper differs from Geanakoplos (2010) in assuming away from risk-neutrality of investors, and analysis on equilibrium with short sales is possible.

Set-up. Timeline. Discrete, finite period t= 0,1,...,T, with two states in each period: Up(U) or Down(D). There is a risky asset, normalized to 1 unit,

- no payoff in intermediate period;
- final payoff: depends on number of ups (and downs)
- asset is traded at each period
- risky-free interest rate is zero

Investors are risk-averse with heterogeneous belief, indexed by h $\in (0, 1)$

- h thinks probability of Up is h.
- h $\in (0,1)$, not $[0,1]$ (otherwise)
- h follows a beta distribution:

$$f(h) = \frac{h^{\alpha-1}(1-h)^{\beta-1}}{B(\alpha,\beta)}, \quad \alpha,\beta > 1$$

Individual Problem. We assume agents behave myopically (no learning), and solve their problems backward. Denote

- begin-of-period wealth: w_h
- price of asset at current period: p
- price of asset at next period: p_d if down, p_u if up;
- unit of asset purchase x_h

Optimization. We consider the problem faced by agent h at any period:

$$\max_{x_h} h \log(w_h - x_h p + x_h p_u) + (1-h)\log(w_h - x_h p + x_h p_d) \tag{433}$$

The first-order condition is

$$x_h = w_h \left(\frac{h}{p - p_d} - \frac{1-h}{p_u - p} \right) \tag{434}$$

- for most pessimistic agent with h $\to 0$,

$$x_h \to w_h \left(-\frac{1-h}{p_u - p} \right) < 0$$

- for most optimistic agent with h $\to 1$,

$$x_h \to \underbrace{w_h}_{wealth} \underbrace{\frac{1}{p - p_d}}_{leverage} > 0$$

- interpretation of leverage: p_d is the max level of borrowing without default risk

Define a risk-neutral probability of up-move: p^*:

$$p = p^* p_u + (1-p^*)p_d$$

Now the first order condition (434) becomes

$$x_h = \frac{w_h}{p_u - p_d} \frac{h - p^*}{p^*(1-p^*)} \tag{435}$$

- for agents with h $> p^*$ (optimistic), they take long position
- for agents with h $< p^*$ (pessimistic), they take short position
- for agents with h $= p^*$ (pessimistic), they take zero position

The realized return on wealth of agent h in next period is linear in his belief:

- up state:

$$w_h + x_h(p_u - p) = w_h + \frac{w_h}{p_u - p_d} \frac{h - p^*}{p^*(1-p^*)}(p_u - p) \equiv w_h \frac{h}{p^*}$$

- down state:

$$w_h - x_h(p - p_d) = w_h + \frac{w_h}{p_u - p_d} \frac{h - p^*}{p^*(1-p^*)}(p - p_d) \equiv w_h \frac{1-h}{1-p^*}$$

Agent h's wealth at current node, given a history of m ups and n downs (history $\{m,n\}$):

$$w_h = \lambda_{path} h^m (1-h)^n$$

where λ_{path} is independent of agent type h.

Equilibrium. Using the condition that aggregate wealth equals total value of asset:

$$\underbrace{\int \lambda_{path} h^m (1-h)^n f(h) dh}_{\text{aggregate wealth}} = \underbrace{p}_{\text{asset value}} \tag{436}$$

We can solve λ_{path} given p as

$$\lambda_{path} = \frac{B(\alpha,\beta)}{B(\alpha+m,\beta+n)} p$$

Now agent h's wealth at current period (state) becomes

$$w_h = \frac{B(\alpha,\beta)}{B(\alpha+m,\beta+n)} h^m (1-h)^n p \tag{437}$$

Combining equation (437) and equation (434), we have

$$x_h = w_h \left(\frac{h}{p-p_d} - \frac{1-h}{p_u - p} \right) = \frac{B(\alpha,\beta)}{B(\alpha+m,\beta+n)} h^m (1-h)^n p \left(\frac{h}{p-p_d} - \frac{1-h}{p_u - p} \right) \tag{438}$$

Using the market clearing condition of risky asset:

$$\int \underbrace{\frac{B(\alpha,\beta)}{B(\alpha+m,\beta+n)} h^m (1-h)^n p \left(\frac{h}{p-p_d} - \frac{1-h}{p_u - p} \right)}_{x_h} \underbrace{\frac{h^{\alpha-1}(1-h)^{\beta-1}}{B(\alpha,\beta)}}_{f(h)} dh = 1 \tag{439}$$

we can solve p as

$$p = \frac{(m+\alpha)p_d p_u + (n+\beta)p_u p_d}{(m+\alpha)p_d + (n+\beta)p_u} \tag{440}$$

The risk-neutral probability (p^*) in equilibrium is solved from $p = p^* p_u + (1-p^*)p_d$:

$$p^* = \frac{(m+\alpha)p_d}{(m+\alpha)p_d + (n+\beta)p_u} \tag{441}$$

Wealth distribution. What is fraction of total wealth owned by type h?

$$\frac{w_h f(h)}{p} = \frac{\frac{B(\alpha,\beta)}{B(\alpha+m,\beta+n)} h^m (1-h)^n p \frac{h^{\alpha-1}(1-h)^{\beta-1}}{B(\alpha,\beta)}}{p} = \frac{h^{m+\alpha-1}(1-h)^{n+\beta-1}}{B(\alpha+m,\beta+n)}$$

- Who has highest level of wealth at history $\{m,n\}$?
  Agent h^* whose beliefs turned out to be most accurate *ex post*:

$$h^* = \frac{m}{m+n}$$

Wealth-weighted Beliefs. Define wealth-weighted cross-sectional average belief, at time t, with history $\{m, n\}$ as $H_{m,t}$:

$$H_{m,t} = \int_0^1 h \frac{w_h f(h)}{p} dh = \frac{m+\alpha}{t+\alpha+\beta} \tag{442}$$

The risk-neutral probability in equilibrium (p^*) in equation (441) can be rewritten as

$$p^* = \frac{H_{m,t} p_d}{H_{m,t} p_d + (1 - H_{m,t})\, p_u} \tag{443}$$

and

$$\frac{p_u}{p} = \frac{H_{m,t}}{p^*} \quad \text{and} \quad \frac{p_d}{p} = \frac{1 - H_{m,t}}{1-p^*} \tag{444}$$

Risk-neutral beliefs (p^*) vs. Wealth-weighted belief ($H_{m,t}$):

- If $H_{m,t} > p^*$: $p_u > p_d$
- If $H_{m,t} < p^*$: $p_u < p_d$
- wealth-weighted average belief ($H_{m,t}$) more optimistic than risk-neutral beliefs (p^*)

Leverage and Portfolio. Share of wealth investor h invests in the risky asset is

$$\frac{x_h p}{w_h} = \frac{h - p^*}{H_{m,t} - p^*}$$

Leverage of investor h, defined as the ratio of funds borrowed to wealth:

$$\frac{x_h p - w_h}{w_h} = \frac{h - H_{m,t}}{H_{m,t} - p^*} \tag{445}$$

Representative investor: investor with $h = H_{m,t}$ can be thought of as the representative agent.

- share of wealth invests in risky asset is 1.

Portfolio management:

- pessimistic investor with $h < p^*$: sell all of the x_h and hold cash
- moderate investor with $h \in (p^*, H_{m,t})$: hold some x_h and some cash
- optimistic investor with $h > H_{m,t}$: leverage to hold all wealth in x_h

Risk Premium. The agent h's (subjectively) perceived excess return is

$$r_h \equiv \frac{hp_u + (1-h)p_d}{p} - 1 = h \underbrace{\frac{p_u}{p}}_{=\frac{H_{m,t}}{p^*}} + (1-h) \underbrace{\frac{p_d}{p}}_{=\frac{1-H_{m,t}}{1-p^*}} - 1 = \frac{(h-p^*)(H_{m,t}-p^*)}{p^*(1-p^*)} \tag{446}$$

To the representative investor with $h = H_{m,t}$, his (subjectively) perceived risk premium is

$$r_{H_{m,t}} \equiv \frac{H_{m,t} p_u + (1 - H_{m,t}) p_d}{p} - 1 = \frac{(H_{m,t} - p^*)(H_{m,t} - p^*)}{p^*(1-p^*)} \tag{447}$$

which coincides with risk-neutral variance of the asset:

$$p^* (\frac{p_u}{p})^2 + (1-p^*)(\frac{p_d}{p})^2 = \frac{(H_{m,t} - p^*)^2}{p^*(1-p^*)} \tag{448}$$

Interpretation of wealth share in risky asset: ratio of perceived excess return to that by representative investor

$$\frac{x_h p}{w_h} = \frac{h - p^*}{H_{m,t} - p^*} = \frac{r_h}{r_{H_{m,t}}}$$

The agent h's perceived variance of asset return is

$$\sigma_h^2 \equiv h(\frac{p_u}{p})^2 + (1-h)(\frac{p_d}{p})^2 - (\frac{hp_u + (1-h)p_d}{p})^2 = \frac{h(1-h)(H_{m,t}-p^*)^2}{p^{*2}(1-p^*)^2} \tag{449}$$

Thus his perceived Sharpe ratio is

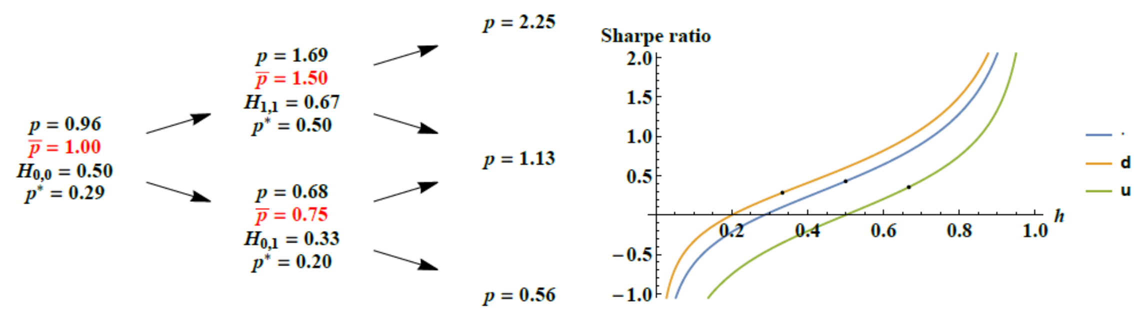
$$\frac{r_h - 0}{\sigma_h} = \frac{\frac{(h-p^*)(H_{m,t}-p^*)}{p^*(1-p^*)}}{\frac{\sqrt{h(1-h)}(H_{m,t}-p^*)}{p^*(1-p^*)}} = \frac{h-p^*}{\sqrt{h(1-h)}}$$

which increases in h.
Heterogeneous-Belief Economy vs Homogeneous Belief Economy.

- homogeneous economy with h =H for all agent:

$$p^* = \frac{Hp_d}{Hp_d + (1-H)\,p_u} \tag{450}$$

- *short-run*: pricing scheme of heterogeneous-belief economy $\sim$ homogeneous-belief economy with $h = H_{m,t}$
- *multi-period*: the similarity breaks as identity of representative investor changes.
  Example 1. Uniformly distributed belief, T =2, and $H_{0,0} = 1/2$.



  - If bad news arrives: money flows to pessimists $\rightarrow$ representative investor's belief or risk-neutral belief is more pessimistic than the homogeneous economy $\rightarrow$ price declines further $\rightarrow$ Sharpe ratio is higher
  - If good news arrives: money flows to optimists $\rightarrow$ representative investor's belief or risk-neutral belief is more optimistic than the homogeneous economy $\rightarrow$ price rises further $\rightarrow$ Sharpe ratio is lower
  - Initial price is lower due to sentiment risk

Result. Recursive Pricing at each node is weighted *harmonic mean* of the next period price, weighted by beliefs of representative investor: Denote the price at period t with m up moves as $p_{m,t}$, and define $z_{m,t} = 1/p_{m,t}$.

$$z_{m,t} = H_{m,t} z_{m+1,t+1} + (1 - H_{m,t}) z_{m,t+1} \tag{451}$$

Initial price can be solved by backward induction given terminal pay-off $p_{m,T}$ (or $z_{m,T}$):

$$z_{0,0} = \sum_{m=0}^{T} c_m z_{m,T} \tag{452}$$

Result: Given terminal payoff level $p_{m;T}$, the initial price is

$$p_0 = \frac{1}{\sum_{m=0}^{T} \frac{c_m}{p_{m,T}}}, \quad \text{where} \quad c_m = \left(\begin{array}{c} T \\ m \end{array} \right) \frac{B(\alpha + m, \beta + T - m)}{B(\alpha, \beta)}$$

Result: (The effect of belief heterogeneity on asset price) If $1/p_{m;T}$ is convex (concave) in m, the price $p_{0,0}$ falls (rises) as heterogeneity increases. Result: Example 1 is an example with $p_{m;T}$ being concave in m so that the asset's price decreases in the degree of belief heterogeneity. (The wisdom of the crowd). Pricing in the heterogeneous-agent economy is identical to pricing in an economy with a representative agent with log utility whose prior belief, as of time 0, about the probability of an up-move has a beta distribution $h0 \sim Beta(\alpha; \beta)$, and who updates his or her beliefs over time via Bayes' rule. Although individual investors do not learn in this limit, last result says that the market exhibits the wisdom of the crowd," in that the redistribution of wealth between agents over time causes the market to behave as if it is learning as a whole about the probability of an up-move.

13.3. * **Simsek (2021).** This note summarizes the review paper by Simsek(2021) on financial speculation driven by belief disagreement (heterogeneity) with a macroeconomic perspective.

Benchmark Model. The model is an infinite-horizon variant of OLG model, such that the old generation never die, but lose some fraction (depreciation) of asset.
technology. A CRS technology:

$$y_t = e_t(n_t + k_t)$$

- aggregate productivity: e_t with growth rate z

$$e_{t+1} = z_{t+1}e_t, \quad z_{t+1} = z \in \{H, L\}$$

- labor: n_t inelastically supplied by the young generation, with wage rate

$$w_t = ne_t$$

- capital: k_t with normalized supply of 1, traded at price P_t with dividend rate

$$r_t = e_t$$

  - the old generation supply $(1-\delta)k_t$
  - the young generation supply δk_t
  - wealth of young generation: $w_t + \delta P_t$
  - return rate to capital is

$$R_{t,t+1} = \frac{(1-\delta)(P_{t+1} + r_{t+1})}{P_t} \tag{453}$$

  - δ fraction depreciates and then injected via young generation in each period
- open economy: access to a foreign asset with constant return R_t^f

optimization. The agents differ in their belief about prospect of the economy $i \in \{o, p\}$: α fraction of the young are optimistic, denoted as type o; $1-\alpha$ fraction are pessimistic, denoted as type p. Each agent i allocates her wealth into consumption and net saving in two types of asset: capital (ω_t^i fraction) and foreign asset ($1-\omega_t^i$ fraction).

$$V_t^i\left(a_t^i\right) = \max_{c_t^i, \omega_t^i} \log c_t^i + \beta E_t^i\left[V_{t+1}^i\left(a_{t+1}^i\right)\right] \tag{454}$$

$$\text{s.t. } a_{t+1}^i = \left(a_t^i - c_t^i\right)\left(R_{t,t+1}\omega_t^i + R_t^f\left(1-\omega_t^i\right)\right) \tag{455}$$

$$\omega_t^i \in [\omega_t, \bar{\omega}_t] \text{ and } a_{t+1}^i \geq 0 \tag{456}$$

The solution to individual portfolio problem is straightforward:

- If $\mathbf{E_t} R_{t,t+1} > R_t^f$, then agents invest in capital as much as possible;
- If $\mathbf{E_t} R_{t,t+1} < R_t^f$, then agents hold risk-less foreign asset as much as possible;
- If $\mathbf{E_t} R_{t,t+1} = R_t^f$, the agent is indifferent.

With log-utility, (individual and aggregate) consumption and saving are linear to (individual and aggregate) wealth:

$$c_t = (1-\beta)a_t^i \quad ; \quad C_t = (1-\beta)\sum_i a_t^i$$

The market clearing conditions of two asset markets are then

$$\sum_i \omega_t^i \beta a_t^i = P_t \quad ; \quad \sum_i (1-\omega_t^i)\beta a_t^i = F_t$$

where F_t denote net holding of foreign asset. The aggregate resource constraint is

$$Y_t = C_t + F_t - R^f_{t-1} F_{t-1} \tag{457}$$

Solving the problem above, we have

$$C_t = (1-\beta)[Y_t + P_t + R^F_{t-1} F_{t-1}] \tag{458}$$

$$P_t + F_t = \beta[Y_t + P_t + R^F_{t-1} F_{t-1}] \tag{459}$$

equilibrium. Given the linearity w.r.t. technology, we characterize an equilibrium that all variables are proportional to e_t. For example, we define $p_t = P_t/e_t$; $c_t = C_t/e_t$; $f_t = F_t/e_t$. Then last two equations imply

$$c_t = (1-\beta)[n + 1 + p + R^F f_{t-1}/z_t] \tag{460}$$

$$p + f_t = \beta[n + 1 + p + R^F f_{t-1}/z_t] \tag{461}$$

which deliver benchmark (BGP) consumption and saving function for given z and p as

$$c = (1-\beta)\frac{1 + n - \left(R^f/z - 1\right) p}{1 - \beta R^f/z} \tag{462}$$

$$f = \frac{\beta(1+n) - (1-\beta)p}{1 - \beta R^f/z} \tag{463}$$

The last step is to solve equilibrium scaled price p given assumption on belief.
common belief. Assume all the agents share the same belief on productivity growth rate $z_{t+1} = z$, either being H or L. Then combining the no-arbitrage condition $R_{t,t+1} = R^F_t$ and equation (453):

$$R_{t,t+1} = R^F = \frac{(1-\delta)[P_{t+1} + e_{t+1}]}{P_t} = \frac{(1-\delta)[p_{t+1} + 1]}{p_t z_{t+1}}, \quad z \in \{H, L\}$$

gives a unique solution of p

$$p(z) = \frac{z(1-\delta)}{R^F - z(1-\delta)} \tag{464}$$

Speculation with Belief Disagreement.
Part 1: Overvaluation. This section shows that the economy with a fraction of optimists can feature asset prices and macroeconomic outcomes as if all investors are optimistic.
Consider a scenario such that

- short-selling is prohibited
- belief is persistent: young optimists(pessimists) become old optimists(pessimists)
- young optimists' total wealth is sufficiently high, i.e. α is large:

$$\alpha\beta(n + \delta p(H)) > p(H)$$

Then there exists an equilibrium with price as H-price equilibrium in common-belief setting:

$$p^* = p(H) = \frac{H(1-\delta)}{R^F - H(1-\delta)} \tag{465}$$

- optimists are indifferent between foreign asset and risky capital at current price

$$E^o_t[R_{t,t+1}] = \frac{(1-\delta)[p(H)He_t + He_t]}{p(H)e_t} = R^F$$

- pessimists strictly prefer foreign asset as their perceived return to risky capital at current price is

$$E_t^p[R_{t,t+1}] = \frac{(1-\delta)[p(H)Le_t + Le_t]}{p(H)e_t} < R^F$$

- pessimists would like to short the asset if allowed

Macroeconomic Implication: Now consider an experiment. Suppose all agents are pessimists ($\alpha = 0$) initially, so that $p = p(L)$ on the BGP. Assume these pessimists have objective belief so that actual realization of growth rate will be $z = L$ for each period t. At period 0, a share of young optimists $\alpha > 0$ with persistent belief are born and their wealth are sufficiently high to push up price to $p(H)$. Discuss the short-run and long-run effects on consumption.

- short-run: asset price $\uparrow \Rightarrow$ *perceived* wealth $(p_0 + f_{-1}) \uparrow \Rightarrow$ consumption $c_0 \uparrow$ & saving in foreign asset $f_0 \downarrow$
- long-run: realized output remain L $\Rightarrow$ wealth $(p_1 + f_0) \downarrow \Rightarrow$ consumption $c_1 \downarrow$

Part 2: Speculative Bubbles. This section further shows that, when a fraction of agents are optimistic, the asset price could exceed the present discounted valuation of all investors in the economy, or *speculative bubble* exists.
Consider a scenario such that

- short-selling is prohibited
- belief is transitory: young optimists(pessimists) become old pessimists(pessimists)
- young optimists' total wealth is sufficiently high, i.e. α is large:

$$\alpha\beta(n + \delta p(H)) > p(H)$$

Then there exists an equilibrium with price as H-price equilibrium in common-belief setting:

$$p^* = p(H) = \frac{H(1-\delta)}{R^F - H(1-\delta)} \tag{466}$$

- optimist's valuation of buy-and-hold is

$$p^o = \frac{H(1-\delta)}{R^F - L(1-\delta)} < p^*$$

- pessimist's valuation of buy-and-hold is

$$p^p = \frac{L(1-\delta)}{R^F - L(1-\delta)} < p^*$$

- price is higher than *present discounted valuation* of all investors in the economy
- price reflects the *resale option value*: young optimists sell the asset to newly-born optimists when they become old pessimist

Part 3: Leveraged Speculative Bubble. So far we have assume that investors have sufficient resources to take their unconstrained optimal portfolio position. This section connects asset price to endogenous collateral constraint arising from heterogeneous belief.
Exogenous Leverage. Consider a scenario such that

- short-selling is prohibited
- belief is transitory: young optimists(pessimists) become old pessimists(pessimists)
- capital does not depreciate: $\delta = 0 \to$ net worth of the young $= n$
- investors face an exogenous leverage constraint:

$$\omega_t^i \leq \bar{\omega}$$

- max leveraged is not sufficiently high:

$$\alpha\beta n\bar{\omega} \in (p(L), p(H)), \text{ where } p(z) = \frac{z(1-\delta)}{R^F - z(1-\delta)}$$

Then there exists an equilibrium with price as :

$$p^* = p(\bar{\omega}) = \alpha\beta n\bar{\omega} \tag{467}$$

- optimist's valuation is higher than current price but constrained by leverage limit
- pessimist's valuation is lower than current price but can't short
- price is increasing with leverage limit $\bar{\omega}$

Endogenous Leverage. We consider a similar setting to the case with leverage limit above, but now arising endogenously. We consider a natural constraint: portfolio return cannot be negative: $R_{t,t+1}\omega_t + R^F(1-\omega_t^i) \geq 0$. This restriction has a natural interpretation:

$$R^F(\omega_t^i - 1) \leq R_{t,t+1}\omega_t \tag{468}$$

The LHS of equation above is total outstanding debt, the RHS is value of asset. Then non-negative return constraint is equivalent to a collateral constraint that rules out default, *in all states* or in *the worst case* from the view of lenders (pessimists). With the worst case being $z_{t+1} = L$, the endogenous leverage limit becomes:

$$\omega_t^i \leq \bar{\omega}^{endo} = \frac{p_t}{p_t - \frac{1}{R^F}L(p_{t+1}+1)} \equiv \frac{P_t}{P_t - P_t(L)}, \text{ where } P_t(L) = \frac{(1-\delta)[P_{t+1,L} + r_{t+1,L}]}{R_t^F} \tag{469}$$

where the denominator is the down-payment, price minus externally financed value, subject to non-default constraint above. As a consequence, an increase in perceived *down-side risk* L will increase leverage, asset price and reduce margin[56].

Then there exists an equilibrium with price as:

$$p = \underbrace{\alpha\beta n}_{net-worth} + \underbrace{\frac{1}{R^F}L(1+p)}_{pledgeable-value} \Rightarrow p = \frac{\alpha\beta n R^F + L}{R^F - L} \tag{471}$$

An Aside: Speculative Bubble vs. Rational Bubble. In this section we compare the rational and speculative mechanism generating asset overvaluation and bubbles.

Rational Bubbles. Consider a scenario such that

- common belief as pessimists: $z = L$
- capital depreciates: $\delta > 0$
- dynamic inefficiency: saving return is no greater than growth rate of the economy

$$R^F = L$$

There exist multiple equilibria, including a continuum of bubbly equilibria:

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[56]Similarly, one can derive an endogenous short-selling limit as

$$\omega_t^i \geq \underline{\omega}^{endo} = -\frac{p_t}{\frac{1}{R^F}H(p_{t+1}+1) - p_t} \equiv -\frac{P_t}{P_t(H) - P_t}, \text{ where } P_t(H) = \frac{(1-\delta)[P_{t+1,H} + r_{t+1,H}]}{R_t^F} \tag{470}$$

The intuition is that, short-sellers are more likely to default when asset price is high, thus the denominator is determined by perceived *up-side risks*, which is H.

- The common-belief equilibrium as equation (464) still holds

$$p^* = p(L) = \frac{L(1-\delta)}{R^F - L(1-\delta)}$$

- A continuum of bubbly equilibria also exist, i.e. investors buy an asset w. no dividend payout at a positive price B_t, as long as $B_t \equiv be_t$ satisfies[57]

$$B_{t+1} = LB_t, \quad \text{or b is a non-negative constant}$$

- consumption and saving respond to asset bubbles

$$c_t = (1-\beta)[1 + n + p(L) + b + \frac{R^F f_{t-1}}{L}]$$

$$p(L) + b + f_t = \beta[1 + n + p(L) + b + \frac{R^F f_{t-1}}{L}]$$

As has been discussed in literature, rational bubbles arising due to dynamic inefficiency improve welfare in the short-run and long-run.

Wealth Dynamics in Speculation Model. This section shows that speculation and short selling generates wealth dynamics that could explain procyclical valuation mechanism.
assumptions. Consider the scenario that

- economy is closed and output is at the level of benchmark model by adjusting risk-free interest rate R_t^F to R_t^{F*} (this assumption excludes foreign asset as state variable):

$$F_t = 0 \quad ; \quad Y_t = Y_t^*$$

- capital depreciate: $\delta \geq 0$
- market is complete: no limit on short selling or leverage
- belief is persistent: young optimists (pessimists) become old optimists (pessimists)

equilibrium. We solve this close-economy model: Due to log-utility, the optimal consumption is linear to wealth

$$Y_t = C_t = (1-\beta)(Y_t + P_t) = (1/\beta - 1)P_t \tag{472}$$

which implies a constant output-asset price ratio as

$$\frac{P_t}{Y_t} = \frac{1}{1/\beta - 1} \tag{473}$$

With production function $Y_t = e_t(n+1)$, the normalized asset price $p_t = P_t/e_t$ is

$$p^* = p_t = \frac{n+1}{1/\beta - 1} \tag{474}$$

Common Belief. We start with the case of common belief, i.e. all investors think $z_{t+1} = z \in \{H, L\}$. The solution to benchmark model (equation 464) still holds in this case.

$$p(z) = \frac{z(1-\delta)}{R_t^{F*} - z(1-\delta)} \tag{475}$$

Combining last two equations solves equilibrium interest rate R_t^{F*}:

$$R_t^{F*} = R_t^F(z) = z(1-\delta)\frac{p*+1}{p^*} = z(1-\delta)\frac{n+1/\beta}{n+1} \tag{476}$$

[57]Common-belief equilibrium is a special case with b =0.

In close-economy with common belief, shifts in common belief doesn't affect normalized asset price but equilibrium interest rate.
Heterogeneous Belief. When the economy is populated of investors with heterogeneous belief, both optimists and pessimists will be at the endogenous limit (equation 469 and 470)in equilibrium. This section shows that asset price reflects a *wealth-weighted average valuation* of investors, and shows how wealth dynamics of each type evolves after realization of growth rate.
Asset Price. We first define wealth share of optimists as (this is different from parameter α, fraction of young optimists)

$$\alpha_t \equiv \frac{\sum_{i=o} a_t^i}{\sum_i a_t^i}$$

Using the result that both optimists and pessimists will be at the endogenous limit:

$$\omega_t^o = \bar{\omega_t}^{endo} = \frac{P_t}{P_t - P_t(L)} \quad ; \quad \omega_t^p = \underline{\omega_t}^{endo} = -\frac{P_t}{P_t(H) - P_t}$$

The market clearing condition for risky asset is

$$\sum_i \omega_t^i \alpha_t^i = \omega_t^o \alpha_t + \omega_t^p (1 - \alpha_t) = 1$$

Combining last two equations, we obtain:

$$\frac{P_t}{P_t - P_t(L)} \alpha_t - \frac{P_t}{P_t(H) - P_t}(1 - \alpha_t) = 1$$

Rearrange these term we obtain:

$$P_t = \left(\alpha_t \frac{1}{P_t(H)} + (1 - \alpha_t) \frac{1}{P_t(L)} \right)^{-1} \tag{477}$$

which is a *wealth-weighted harmonic average* of optimists' and pessimists' valuations.
Wealth Dynamics. Following the property that both optimists and pessimists will be at the endogenous limit, we have a straightforward and extreme result that when the state optimists (pessimists) bet doesn't realize, vintage optimists' (pessimists') wealth becomes zero (but there will be young generation born). This implies optimists' wealth share is

$$\alpha_{t,z} = \begin{cases} \underbrace{\dfrac{\alpha(w_t + \delta P_t)}{w_t + r_t + P_t}}_{\text{young optimists}} = \frac{\alpha(n+\delta p^*)}{n+1+p^*}, & if \quad z = L \\ 1 - \underbrace{\dfrac{(1-\alpha)(n + \delta p^*)}{n + 1 + p^*}}_{\text{young pessimists}}, & if \quad z = H \end{cases} \tag{478}$$

This result confirms that $\alpha_{t,L} < \alpha_{t,H}$: optimists become more dominant after the realization of good states, whereas pessimists become more dominant in bad states.
Interest Rate. We check the procyclicality of interest rate. Recall that

$$P_t(H) = \frac{(1-\delta)[P_{t+1,H} + r_{t+1,H}]}{R_t^F} = \frac{(1-\delta)[p^* + 1]He_t}{R_t^F}; P_t(L) = \frac{(1-\delta)[p^* + 1]Le_t}{R_t^F}$$

Plug last equation into equation (477), we can solve equilibrium interest rate as

$$P_t \equiv p^* e_t = \left(\alpha_{t,z} \frac{R_t^F}{(1-\delta)[p^* + 1]He_t} + (1 - \alpha_{t,z}) \frac{R_t^F}{(1-\delta)[p^* + 1]Le_t} \right)^{-1} \tag{479}$$

Note that in the last equation, p^* is independent of z. Equilibrium interest rate depends on recent realization of growth rate as wealth share $\alpha_{t,z}$, and it's procyclical.

13.4. *** Dong, Liu, Wang and Zha (2022).** This paper provides a micro-foundation for housing demand shock that are often introduced in reduced-form as main driving force for housing price fluctuations, provides an explanation for the observed large fluctuations in the price-to-rent ratio that comoves housing prices.

Rep-Agent Model and Price-to-Rent Puzzle. This section presents a stylized representative-agent model to illustrate the role of housing demand shocks in driving the house price, and highlight a price-to-rent puzzle.

The endowment economy has one unit of housing supply and an exogenous endowment of y_t units of consumption goods. The representative household maximize

$$\mathbb{E}\sum_{t=0}^{\infty}\beta^t\left\{\log c_t+\varphi_t\frac{h_t^{1-\theta}}{1-\theta}\right\}$$

subject to the flow of funds constraint

$$c_t+q_t\left(h_t-h_{t-1}\right)\leq y_t+\frac{b_t}{R_t}-b_{t-1}$$

where c_t denotes consumption; h_t denotes housing; φ_t denotes a housing demand shock. The parameter $\beta\in(0,1)$ is the subjective discount factor and $\theta>0$ is a parameter that measures the curvature of the utility function with respect to housing.
Euler equation for housing:

$$\frac{q_t}{c_t}=\beta\mathbb{E}_t\frac{q_{t+1}}{c_{t+1}}+\varphi_t h_t^{-\theta}$$

Euler equation for bond holdings

$$1=\beta R_t\mathbb{E}_t\frac{c_t}{c_{t+1}}$$

Market clearing conditions:

$$c_t=y_t$$

$$b_t=0$$

$$h_t=1$$

House pricing equation: (iterating the housing Euler equation forward, plus the goods and housing market clearing conditions)

$$\frac{q_t}{y_t}=\beta\mathbb{E}_t q_{t+1}\frac{1}{y_{t+1}}+\varphi_t$$

or equivalently,

$$q_t=y_t\left[\mathbb{E}_t\sum_{j=0}^{\infty}\beta^j\varphi_{t+j}\right]$$

Implicit (or shadow) rent: (given by the household's marginal rate of substitution between housing and non-housing consumption)

$$r_{ht}=\varphi_t y_t$$

Price-to-rent ratio:

$$\frac{q_t}{r_{ht}}=\frac{1}{\varphi_t}\mathbb{E}_t\sum_{j=0}^{\infty}\beta^j\varphi_{t+j}$$

price-to-rent puzzle. Assume a stationary process for the housing demand shock

$$\hat{\varphi}_t = \rho\hat{\varphi}_{t-1} + e_t,$$

Log-linearized house price equation:

$$\hat{q}_t = \hat{y}_t + (1-\beta)\mathbb{E}_t\left[\sum_{j=0}^{\infty}\beta^j\hat{\varphi}_{t+j}\right] = \hat{y}_t + \frac{1-\beta}{1-\beta\rho}\hat{\varphi}_t$$

Log-linearized the rent equation:

$$\hat{r}_{ht} = \hat{y}_t + \hat{\varphi}_t$$

Log-linearized price-to-rent ratio:

$$\hat{q}_t - \hat{r}_{ht} = -\frac{\beta(1-\rho)}{1-\beta\rho}\hat{\varphi}_t$$

There are two counter-factual implications of this representative agent model: (1) The model implies that the price-to-rent ratio falls when house price rises, while the price-rent ratio are highly positively correlated with house prices in the data. (2) Second, The model implies that the house price is less volatile than the rent (assume that the endowment is constant so that $\hat{y}_t = 0$), while the opposite is true in the data.

$$\frac{\mathbb{STD}\left(\hat{q}_t\right)}{\mathbb{STD}\left(\hat{r}_{ht}\right)} = \frac{1-\beta}{1-\beta\rho} < 1$$

Hetero-Agent Model with Heterogeneous Belief in Growth Rate of Output. We now present a microeconomic foundation for the housing demand shock by incorporating household-level heterogeneity in the model. We show that this heterogeneous-agent model can generate large volatility in both housing prices and the price-to-rent ratio, and is thus able to resolve the price-rent puzzle. In particular, we consider family members have heterogeneous belief about future housing prices. We model heterogeneous belief as follows.

The aggregate output

$$\frac{y_{t+1}}{y_t} = g_{t+1}, \tag{480}$$

where g_{t+1} is i.i.d. distributed according to a distribution $\tilde{F}$. Each member in period t, however has, different belief of g_{t+1}. In particular, we assume member j believes $g_{t+1} = e_t^j$, where e_t^j is i.i.d. and distributed according to F. Notice that $\tilde{F}$ and F may not be the same. The household problem now simplifies to

$$\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left[\log c_t + \varphi\frac{s_{ht}^{1-\theta}}{1-\theta}\right],$$

with the constraint

$$c_t + R_{ht}s_{ht} + a_t = y_t + (Q_t + R_{ht})\int h_{t-1}(e_{t-1})dF(e_{t-1}) - \int b_{t-1}(e)dF(e_{t-1}), \tag{481}$$

where $h_{t-1}(e_{t-1})$ and $b_{t-1}(e_{t-1})$ denotes house holding and bond holding of the family member who had idiosyncratic belief shock e_{t-1}. The household owns $\int h_{t-1}(e_{t-1})dF(e_{t-1})$ unit of houses in total, and each units can be sold at price Q_t and receives r_{ht} rent incomes, and needs to pay $\int b_{t-1}(e)dF(e_{t-1})$ debt in total, and receive income y_t. This explains the right hand side of equation (481).

In the decentralized housing markets, the household member with belief shock e_t finances house purchases with both family transfer a_t and external debt $b_t(e_t)$, subject to three

constraints:
(i) the flow-of-funds constraint

$$Q_t h_t(e_t) \leq a_t + \frac{b_t(e_t)}{R_t}, \tag{482}$$

(ii) borrowing constraint

$$\frac{b_t(e_t)}{R_t} \leq \kappa_t Q_t h_t(e_t), \tag{483}$$

(iii) non-negative house holding constraint (no short-selling)

$$h_t(e_t) \geq 0. \tag{484}$$

Denote by $\lambda_t, \eta_t(e_t)$, $\pi_t(e_t)$, and $\mu_t(e_t)$ the Lagrangian multiplers associated with the constraints (481), (482), (**??**), and (484), respectively.
FOC w.r.t c_t:

$$\frac{1}{c_t} = \lambda_t. \tag{485}$$

FOC w.r.t s_{ht}:

$$\lambda_t R_{ht} = \varphi s_{ht}^{-\theta}. \tag{486}$$

FOC w.r.t a_t:

$$\lambda_t = \int \eta_t(e_t) dF(e_t). \tag{487}$$

FOC w.r.t $h_t(e_t)$:

$$\eta_t(e_t) Q_t = \beta \mathbb{E}_t \left\{ \lambda_{t+1} \left[Q_{t+1} + R_{ht+1} \right] | \frac{y_{t+1}}{y_t} = e_t \right\} + \kappa_t Q_t \pi_t(e_t) + \mu_t(e_t). \tag{488}$$

Or

$$\eta_t(e_t) q_t y_t = \beta \mathbb{E}_t \frac{1}{y_{t+1}} y_{t+1} [q_{t+1} + r_{ht+1}] + \kappa_t Q_t \pi_t(e_t) + \mu_t(e_t). \tag{489}$$

FOC w.r.t $b_t(e_t)$:

$$\begin{aligned} \eta_t(e_t) &= \beta R_t \mathbb{E}_t \left[\lambda_{t+1} | \frac{y_{t+1}}{y_t} = e_t \right] + \pi_t(e_t) \\ &= \beta R_t \frac{1}{y_t e_t} + \pi_t(e_t). \end{aligned} \tag{490}$$

Market clearing conditions:

$$c_t = y_t$$
$$s_{ht} = \int h_{t-1}(e_{t-1}) dF(e_{t-1})$$
$$\int h_t(e_t) dF(\varepsilon_t) = 1$$
$$\int b_t(e_t) dF(e_t) = 0.$$

Notice since $\int h_{t-1}(e_{t-1}) dF(e_{t-1}) = 1$, we then have $s_{ht} = 1$.

We conjecture the equilibrium price $Q_t = q(\kappa_t) y_t \equiv q_t y_t$, $R_{ht} = r_h(\kappa_t) y_t \equiv r_{ht} y_t$, and $R_t = R(\kappa_t)$. Equation (486) implies that

$$R_{ht} = \varphi y_t. \tag{491}$$

Notice the return of housing between period t to period t+1 is given by

$$\frac{Q_{t+1}+R_{ht+1}}{Q_t}=\frac{q_{t+1}+r_{ht+1}}{q_t}\frac{y_{t+1}}{y_t},$$

So if agent belief that $\frac{y_{t+1}}{y_t}=e_t$, then he will perceive the expected return for housing is

$$\mathbb{E}_t\left[\frac{Q_{t+1}+R_{ht+1}}{Q_t}|\frac{y_{t+1}}{y_t}=e_t\right]=\mathbb{E}_t\left[\frac{q_{t+1}+r_{ht+1}}{q_t}\frac{y_{t+1}}{y_t}|\frac{y_{t+1}}{y_t}=e_t\right]=e_t\mathbb{E}_t\left[\frac{q_{t+1}+r_{ht+1}}{q_t}\right]$$

The perceived housing return is higher for members with higher e_t. The return for bonds is R_t, which is the same for all member. So in equilibrium, there must exist a cutoff e_t^*, such that member will purchase housing if and only if $e_t\geq e_t^*$.

For the marginal trader, $\pi_t(e_t^*)=\mu_t(e_t^*)=0$, then (490) imply that

$$\eta_t(e_t^*)=\beta R_t\left[\frac{1}{y_te_t^*}\right],\tag{492}$$

and equation (489) implies

$$\beta R_t\left[\frac{1}{y_te_t^*}\right]q_ty_t=\beta\mathbb{E}_t\left\{[q_{t+1}+r_{ht+1}]\right\},\tag{493}$$

Or we have

$$q_t=\frac{e_t^*}{R_t}\mathbb{E}_t\left[q_{t+1}+r_{ht+1}\right].\tag{494}$$

We now consider two cases.

(1) First we consider $e_t\geq e_t^*$, we have $\pi_t(e_t)>0$, hence we have constraint (483) binds and hence $h_t(e_t)=\frac{1}{1-\kappa_t}$. This implies that $\mu_t(e_t)=0$. Equations (489) and (490) together implies that

$$\pi_t(e_t)=\frac{\beta}{(1-\kappa_t)\,y_tq_t}\left[\mathbb{E}_t(q_{t+1}+r_{ht+1})-R_t\frac{q_t}{e_t}\right],\tag{495}$$

and

$$\begin{aligned}\eta_t(e_t) &= \beta R_t\frac{1}{y_te_t}+\frac{\beta}{(1-\kappa_t)\,y_tq_t}\left[\mathbb{E}_t\left[q_{t+1}+\varphi\right]-R_t\frac{q_t}{e_t}\right]. \\ &= \beta R_t\frac{1}{y_te_t}+\frac{\beta R_tq_t}{(1-\kappa_t)\,y_tq_t}[\frac{1}{e_t^*}-\frac{1}{e_t}]\end{aligned}\tag{496}$$

where the second line has used the fact $\mathbb{E}_t(q_{t+1}+r_{ht+1})=R_t\frac{q_t}{e_t^*}$.

(2) In the other case $\pi_t(e_t)=0$, equation (490) then implies that

$$\eta_t(e_t)=\beta R_t\frac{1}{y_te_t},\tag{497}$$

then equation (489) implies that

$$\begin{aligned}\mu_t(e_t) &= \beta R_t\frac{1}{e_t}q_t-\beta\mathbb{E}_t[q_{t+1}+r_{ht+1}] \\ &= \beta R_tq_t(\frac{1}{e_t}-\frac{1}{e_t^*})>0\end{aligned}\tag{498}$$

Again the second has used the fact $\mathbb{E}_t(q_{t+1}+r_{ht+1})=R_t\frac{q_t}{e_t^*}$. Since $\mu_t(e_t)h_t(e_t)=0$. This implies $h_t(e_t)=0$.

With the expression of $\eta_t(e_t)$, we can rewrite equation (487) as

$$\frac{1}{c_t} = \beta R_t \frac{1}{y_t}\int \frac{1}{e}dF(e) + \frac{\beta R_t}{(1-\kappa_t)\,y_t}\int_{e_t^*}[\frac{1}{e_t^*} - \frac{1}{e}]dF(e),$$

or

$$1 = \beta R_t \int_{e_{\min}}^{e_{\max}} \frac{1}{e}dF(e) + \frac{\beta R_t}{(1-\kappa_t)}\int_{e_t^*}[\frac{1}{e_t^*} - \frac{1}{e}]dF(\varepsilon), \tag{499}$$

Finally housing market clearing condition yields

$$\frac{1}{1-\kappa_t}\int_{e_t^*}^{\varepsilon_{\max}} dF(\varepsilon) = 1. \tag{500}$$

We then have

$$\begin{aligned} \frac{e_t^*}{R_t} &= \beta e_t^* \int \frac{1}{e}dF(e) + \frac{\beta}{(1-\kappa_t)}\int_{e_t^*}[1 - \frac{e_t^*}{e}]dF(e) \\ &= \beta e_t^* \int \frac{1}{e}dF(e) + \frac{\beta}{1-F(e_t^*)}\int_{e_t^*}[1 - \frac{e_t^*}{e}]dF(e) \\ &= \beta + \beta e_t^*[\int \frac{1}{e}dF(\varepsilon) - \frac{1}{1-F(e_t^*)}\int_{e_t^*}^{e_{\max}} \frac{1}{e}dF(e)]. \end{aligned} \tag{501}$$

Denote $\phi(e_t^*) = \frac{1}{1-F(e_t^*)}\int_{e_t^*}^{e_{\max}} \frac{1}{e}dF(e)$, we have

$$\begin{aligned} \frac{\phi'(e_t^*)}{\phi(e_t^*)} &= \frac{f(e_t^*)}{1-F(e_t^*)} - \frac{f(e_t^*)}{\int_{e_t^*}^{e_{\max}} \frac{e_t^*}{e}dF(e)} \\ &< \frac{f(e_t^*)}{1-F(e_t^*)} - \frac{f(e_t^*)}{\int_{e_t^*}^{e_{\max}} 1dF(e)} \\ &= \frac{f(e_t^*)}{1-F(e_t^*)} - \frac{f(e_t^*)}{1-F(e_t^*)} = 0 \end{aligned} \tag{502}$$

Hence we have $\phi'(e_t^*)$. It is then obvious that

$$\frac{\partial\,(e_t^*/R_t)}{\partial e_t^*} > 0 \tag{503}$$

Finally it is easy to see $\frac{\partial(e_t^*)}{\partial \kappa_t} > 0$. Notice the price rent ratio $Q_t/R_{ht} = q_t/r_{ht} = q_t/\varphi$.

In summary we then have the following proposition.

Proposition 13.1. *when κ_t increases, e_t^* increases, $\frac{e_t^*}{R_t}$ increases, housing price Q_t increases, the price rent ratio, $q_t = Q_t/R_{ht}$ increases, the rents R_{ht} remain constant in the steady state.*

Micro-foundation for housing demand shock: credit supply. We now consider that each agents can obtain credit randomly with probability p_t. Let $z_t^j = 1$ indexes households who can obtain credit and $z_t^j = 0$ indexes household who can not obtain credit. Then the budget constraint becomes

$$\begin{gathered} c_t + R_{ht}s_{ht} + a_t = \\ y_t + (Q_t + R_{ht})\left[p_{t-1}\int h_{t-1}(e_{t-1},1)dF(e_{t-1}) + (1-p_{t-1})\int h_{t-1}(e_{t-1},0)dF(e_{t-1})\right] \\ -p_{t-1}\int b_{t-1}(e,1)dF(e_{t-1}) - (1-p_{t-1})\int b_{t-1}(e,0)dF(e_{t-1}), \end{gathered}$$

And in the decentralized housing markets, the household member with belief shock e_t finances house purchases with both family transfer a_t and external debt $b_t(e_t, 1)$ if he can obtain credit, subject to the flow-of-funds constraint

$$Q_t h_t(e_t, 1) \leq a_t + \frac{b_t(e_t, 1)}{R_t}, \tag{504}$$

and the borrowing constraint

$$\frac{b_t(e_t, 1)}{R_t} \leq \kappa Q_t h_t(e_t, 1), \tag{505}$$

where, as in the benchmark model, the risk-free interest rate R_t and the loan-to-value ratio κ are common for all borrowers. In addition, the housing purchase must be non-negative, namely

$$h_t(e_t) \geq 0. \tag{506}$$

And a household flow-of-funds constraint is

$$Q_t h_t(e_t, 0) \leq a_t + \frac{b_t(e_t, 0)}{R_t}, \tag{507}$$

and the borrowing constraint

$$\frac{b_t(e_t, 0)}{R_t} \leq 0, \tag{508}$$

The first order condition with respect to a_t (487) becomes

$$\lambda_t = p_t \int \eta_t(e_t, 1) dF(e_t) + (1 - p_t) \int \eta_t(e_t, 0) dF(e_t). \tag{509}$$

Notice since agents with acess to credit market or not draw the belief about next period growth rate from the same distribution. Then there will exist a same cutoff e_t^* defined as

$$q_t = \frac{\varepsilon_t^*}{R_t} E_t[q_{t+1} + r_{ht+1}], \tag{510}$$

we have

$$h_t(e_t, 1) = \left\{ \begin{array}{cc} \frac{1}{1-\kappa} & \text{if } e_t \geq e_t^* \\ 0 & \text{otherwise} \end{array} \right\} \tag{511}$$

and

$$h_t(e_t, 0) = \left\{ \begin{array}{cc} 1 & \text{if } e_t \geq e_t^* \\ 0 & \text{otherwise} \end{array} \right\} \tag{512}$$

The first order condition for $h_t(e_t, 1)/h_t(e_t, 0)$ and $b_t(e_t, 1)/b_t(e_t, 0)$ are given by

$$\eta_t(e_t, 1) q_t y_t = \beta \mathbb{E}_t \frac{1}{y_{t+1}} y_{t+1} [q_{t+1} + r_{ht+1}] + \kappa Q_t \pi_t(e_t, 1) + \mu_t(e_t, 1). \tag{513}$$

$$\eta_t(e_t, 0) q_t y_t = \beta \mathbb{E}_t \frac{1}{y_{t+1}} y_{t+1} [q_{t+1} + r_{ht+1}] + \mu_t(e_t, 0). \tag{514}$$

The first order condition with respect to $b_t(e_t, 1)$ is

$$\begin{aligned} \eta_t(e_t, 1) &= \beta R_t \mathbb{E}_t \left[\lambda_{t+1} | \frac{y_{t+1}}{y_t} = e_t \right] + \pi_t(e_t, 1) \\ &= \beta R_t \frac{1}{y_t e_t} + \pi_t(e_t, 1). \end{aligned} \tag{515}$$

The first order condition with respect to $b_t(e_t, 0)$ is

$$\begin{aligned}\eta_t(e_t, 0) &= \beta R_t \mathbb{E}_t\left[\lambda_{t+1}|\frac{y_{t+1}}{y_t} = e_t\right] + \pi_t(e_t, 1) \\ &= \beta R_t \frac{1}{y_t e_t} + \pi_t(e_t, 0).\end{aligned} \tag{516}$$

Similary the benchmark case we have

$$\begin{aligned}\eta_t(e_t, 1) &= \beta R_t \frac{1}{y_t e_t} + \beta \frac{R_t}{1-\kappa} \frac{1}{y_t} \max[\frac{1}{e_t^*} - \frac{1}{e_t}, 0] \\ \eta_t(e_t, 0) &= \beta R_t \frac{1}{y_t e_t} + \beta R_t \frac{1}{y_t} \max[\frac{1}{e_t^*} - \frac{1}{e_t}, 0]\end{aligned}$$

This then implies that equation (509) becomes

$$\frac{1}{c_t} = \beta R_t \int \frac{1}{y_t e_t} dF(e_t) + \left[\frac{p_t}{1-\kappa} + (1-p_t)\right] \frac{\beta R_t}{y_t} \int_{e_t^*} [\frac{1}{e_t^*} - \frac{1}{e_t}] dF(e_t)$$

or

$$1 = \beta R_t \int \frac{1}{e} dF(e) + \beta R_t \left[\frac{p_t}{1-\kappa} + (1-p_t)\right] \int_{e_t^*} [\frac{1}{e_t^*} - \frac{1}{e_t}] dF(e_t)$$

Finally the housing clearing becomes

$$\frac{p_t}{1-\kappa} \int_{e_t^*}^{e_{\max}} dF(\varepsilon) + (1-p_t) \int_{e_t^*}^{e_{\max}} dF(\varepsilon) = 1$$

Or we have

$$\frac{p_t}{1-\kappa} + (1-p_t) = \frac{1}{1-F(\varepsilon_t^*)} \tag{517}$$

This in turn means that

$$\begin{aligned}\frac{e_t^*}{R_t} &= \beta e_t^* \int \frac{1}{e} dF(e) + \beta \left[\frac{p_t}{1-\kappa} + (1-p_t)\right] \int_{e_t^*} [1 - \frac{e_t^*}{e}] dF(e) \\ &= \beta e_t^* \int \frac{1}{e} dF(e) + \frac{\beta}{1-F(e_t^*)} \int_{e_t^*} [1 - \frac{e_t^*}{e}] dF(e) \\ &= \beta + \beta e_t^* [\int \frac{1}{e} dF(\varepsilon) - \frac{1}{1-F(e_t^*)} \int_{e_t^*}^{e_{\max}} \frac{1}{e} dF(e)].\end{aligned} \tag{518}$$

Again we have $\frac{e_t^*}{R_t}$ increases with e_t^*. Notice $\frac{\partial \varepsilon_t^*}{\partial p_t} > 0$. We then have a similar proposition.

Proposition 13.2. *when p_t increases, e_t^* increases, $\frac{e_t^*}{R_t}$ increases, housing price Q_t increases, the price rent ratio, $q_t = Q_t/R_{ht}$ increases, the rents R_{ht} remain constant in the steady state.*

Notice in this case the loan to value ratio does not change. But more agents are able to borrow.

14. Asset Bubble

14.1. **Abreu and Brunnermeier (2003, ECMA).** This paper studies how *asynchronous awareness* breaks common knowledge and how this in turn may cause significant delay in the response of equilibrium outcomes to changes in the environment. An application of their framework shows asynchronous awareness delays the burst of a bubble.

14.2. *** Miao and Wang (2018, AER).** This paper provides a theory of stock price bubbles in the presence of endogenous credit constraints in production economies with infinitely-lived agents.

Setting. Consider a variant of Miao and Wang (2018). Consider a continuum of firms endowed with technology:

$$Y_{jt} = AK_{jt}^{\alpha} N_{jt}^{1-\alpha} \tag{519}$$

In each period, the firms solve the following Bellman equation:

$$V_t(K_{jt}, L_{jt}, \varepsilon_{jt}) = \max_{N_j t} AK_{jt}^{\alpha} N_{jt}^{1-\alpha} - W_t N_{jt} + \frac{L_{jt+1}}{R_{ft}} - L_{jt} - I_{jt} + \beta E_t V_{t+1}(K_{jt+1}, L_{jt+1}, \varepsilon_{jt+1})$$

And the capital law of motion

$$K_{jt+1} = (1-\delta)K_{jt} + I_{jt}\varepsilon_{jt}$$

is subject to investment efficiency shock: ε_{jt} that follows distribution function $1-F(\varepsilon) = \varepsilon^{-\sigma}$. The firm also faces a no-equity finance constraint:

$$d_{jt} \equiv AK_{jt}^{\alpha} N_{jt}^{1-\alpha} - W_t N_t + \frac{L_{jt+1}}{R_{ft}} - L_{jt} - I_{jt} \geq 0$$

Finally, there is a borrowing constraint:

$$\frac{L_{jt+1}}{R_{ft}} \leq \beta E_t V_{t+1}(\xi K_{jt}, 0, \varepsilon_{jt+1})$$

We first solve the static labor policy function:

$$(1-\alpha)\frac{Y_{jt}}{N_{jt}} = W_t$$

which implies flow profit as

$$\max_{N_j t} AK_{jt}^{\alpha} N_{jt}^{1-\alpha} - W_t N_{jt} = \alpha A_t [\frac{(1-\alpha)A}{W}]^{\frac{1-\alpha}{\alpha}} K_{jt} \equiv R_t K_{jt} \tag{520}$$

Policy function. We start with the individual firm's problem. Conjecture that: (1). firm value is given by $\beta E_t V_{t+1}(K_{jt+1}, L_{jt+1}, \varepsilon_{jt+1}) = Q_t K_{jt+1} - \frac{L_{jt+1}}{R_{ft}} + B_t$. (2) There exists a cut-off ε_t^* such that firms with $\varepsilon_{jt} > \varepsilon_t^*$ invest and don't pay out dividend. We can re-write the problem as

$$V_{jt}(K_{jt}, L_{jt}, \varepsilon_{jt}) = \max_{I_{jt}} [R_t + Q_t(1-\delta)]K_{jt} - L_{jt} + (Q_t\varepsilon_{jt} - 1)I_{jt} + B_t \tag{521}$$

It's straightforward that the cut-off productivity is given by

$$Q_t \varepsilon_t^* - 1 = 0, \quad or \quad \varepsilon_t^* = \frac{1}{Q_t} \tag{522}$$

Now we solve the individual problem.

- For firms beyond cut-off (i.e. $\varepsilon_{jt} > \varepsilon_t^*$): they do not pay out dividend and invest as much as possible, thus

$$d_{jt} \equiv R_t K_{jt} + \frac{L_{jt+1}}{R_{ft}} - L_{jt} - I_{jt} = 0 \tag{523}$$

and

$$\frac{L_{jt+1}}{R_{ft}} = \beta E_t V_{t+1}(\xi K_{jt}, 0, \varepsilon_{jt+1}) \tag{524}$$

must hold in equality, which implies that borrowing and investment are

$$L_{jt+1} = R_{ft}\beta E_t V_{t+1}(\xi K_{jt}, 0, \varepsilon_{jt+1})$$

and

$$I_{jt} = R_t K_{jt} + \beta E_t V_{t+1}(\xi K_{jt}, 0, \varepsilon_{jt+1}) - L_{jt}$$

Given our conjecture on the form of value function, such that

$$\beta E_t V_{t+1}(K_{jt+1}, L_{jt+1}, \varepsilon_{jt+1}) = Q_t K_{jt+1} - \frac{L_{jt+1}}{R_{ft}} + B_t \tag{525}$$

we have then

$$\frac{L_{jt+1}}{R_{ft}} = \xi Q_t K_{jt} + B_t \tag{526}$$

and

$$I_{jt} = (R_t + \xi Q_t)K_{jt} + B_t - L_{jt} \tag{527}$$

Therefore,

$$\begin{aligned} V_{jt}(K_{jt}, L_{jt}, \varepsilon_{jt}) &= [R_t + Q_t(1-\delta)]K_{jt} - L_{jt} + (Q_t\varepsilon_{jt} - 1)[(R_t + \xi Q_t)K_{jt} + B_t - L_{jt}] + B_t \\ &= [R_t + Q_t(1-\delta) + (Q_t\varepsilon_{jt} - 1)(R_t + \xi Q_t)]K_{jt} - Q_t\varepsilon_{jt}L_{jt} + Q_t\varepsilon_{jt}B_t, \quad \forall \varepsilon_{jt} \geq \varepsilon_t^* \end{aligned} \tag{528}$$

- For firms below cut-off (i.e. $\varepsilon_{jt} < \varepsilon_t^*$): they do not invest, and pay out dividend.

$$V_{jt}(K_{jt}, L_{jt}, \varepsilon_{jt}) = [R_t + Q_t(1-\delta)]K_{jt} - L_{jt} + B_t, \quad \forall \varepsilon_{jt} \leq \varepsilon_t^* \tag{529}$$

Verifying Conjecture. Now we verify our conjecture on form of value function.

$$\begin{aligned} &\beta E_t V_{t+1}(K_{jt+1}, L_{jt+1}, \varepsilon_{jt+1}) = \beta \int^{\varepsilon_t^*} [R_{t+1} + Q_{t+1}(1-\delta)]K_{jt+1} - L_{jt+1} + B_{t+1} dF(\varepsilon) \\ &+\beta \int_{\varepsilon_t^*} [R_{t+1} + Q_{t+1}(1-\delta) + (Q_{t+1}\varepsilon - 1)(R_{t+1} + \xi Q_{t+1})]K_{jt+1} - Q_{t+1}\varepsilon L_{jt+1} + Q_{t+1}\varepsilon B_{t+1} dF(\varepsilon) \\ = \quad &\beta \left\{[R_{t+1} + Q_{t+1}(1-\delta)]K_{jt+1} - L_{jt+1} + B_{t+1}\right\} F(\varepsilon_t^*) \\ &+\beta[R_{t+1} + Q_{t+1}(1-\delta) - (R_{t+1} + \xi Q_{t+1})]K_{jt+1}[1 - F(\varepsilon_t^*)] \\ &+\beta\{(R_{t+1} + \xi Q_{t+1})Q_{t+1}K_{jt+1} - Q_{t+1}L_{jt+1} + Q_{t+1}B_{t+1}\} \int_{\varepsilon_t^*} \varepsilon dF(\varepsilon) \\ = \quad &\beta[R_{t+1} + Q_{t+1}(1-\delta)]K_{jt+1} + \beta\left\{-L_{jt+1} + B_{t+1}\right\} F(\varepsilon_t^*) - \beta(R_{t+1} + \xi Q_{t+1})K_{jt+1}[1 - F(\varepsilon_t^*)] \\ &+\beta\{(R_{t+1} + \xi Q_{t+1})Q_{t+1}K_{jt+1} - Q_{t+1}L_{jt+1} + Q_{t+1}B_{t+1}\} \int_{\varepsilon_t^*} \varepsilon dF(\varepsilon) \end{aligned} \tag{530}$$

which must be consistent with

$$\beta E_t V_{t+1}(K_{jt+1}, L_{jt+1}, \varepsilon_{jt+1}) = Q_t K_{jt+1} - \frac{L_{jt+1}}{R_{ft}} + B_t \tag{531}$$

Combining last two equations solve recursive formulas for Q_t, R_{ft} and B_t respectively. (I don't want to solve integral QaQ)

Equilibrium condition. To characterize the aggregate dynamics, we need to impose the following equilibrium conditions:

- given exogenous labor supply (N^s), labor market clearing condition solves eqm W_t (and thus R_t):

$$\int_{\varepsilon_t^*} N_{jt} dF(\varepsilon) = N^s \tag{532}$$

- aggregate capital law of motion

$$K_{t+1} = (1-\delta)K_t + I_t = (1-\delta)\int K_{jt} dF(\varepsilon) + \int_{\varepsilon_t^*} I_{jt}\varepsilon dF(\varepsilon)$$

  or equivalently

$$K_{t+1} = \left[(1-\delta) + (R_t + Q_t\xi)\frac{\sigma}{\sigma-1}Q_t^{\sigma-1}\right]K_t + B_t\frac{\sigma}{\sigma-1}Q_t^{\sigma-1} \tag{533}$$

- debt market clearing condition

$$L_{t+1} = \int L_{jt+1} dF(\varepsilon) = 0 \tag{534}$$

Equilibrium. Last three equations and formulas for Q_t, R_{ft} and B_t fully characterize aggregate sequence of $\{W_t, K_t, L_t, B_t, Q_t, R_{ft}\}$.
Existence of Bubbly Equilibrium. The three pricing equations are

$$Q_t = \beta\left[\left(1 + \frac{Q_{t+1}^{\sigma}}{\sigma-1}\right)R_{t+1} + Q_{t+1}(1-\delta) + \frac{Q_{t+1}^{\sigma+1}\xi}{\sigma-1}\right] \tag{535}$$

$$B_t = \beta\left(1 + \frac{Q_{t+1}^{\sigma}}{\sigma-1}\right)B_{t+1} \tag{536}$$

$$\frac{1}{R_{ft}} = \beta\left(1 + \frac{Q_{t+1}^{\sigma}}{\sigma-1}\right) \tag{537}$$

Suppose there exists a bubbly equilibrium with $B_t > 0$, then

$$K = \left[(1-\delta) + (R + Q\xi)\frac{\sigma}{\sigma-1}Q^{\sigma-1}\right]K + B\frac{\sigma}{\sigma-1}Q^{\sigma-1} > \left[(1-\delta) + (R + Q\xi)\frac{\sigma}{\sigma-1}Q^{\sigma-1}\right]K \tag{538}$$

Formula of B_t implies

$$\beta\left(1 + \frac{Q^{\sigma}}{\sigma-1}\right) = 1$$

or

$$Q = [(\frac{1}{\beta} - 1)(\sigma-1)]^{1/\sigma} \tag{539}$$

Plug this into formula of Q_t:

$$Q = \beta\left[\left(1 + \frac{Q^{\sigma}}{\sigma-1}\right)R + Q(1-\delta) + \frac{Q^{\sigma+1}\xi}{\sigma-1}\right]$$

we obtain that

$$R = Q[1 - \beta(1-\delta) - \xi(1-\beta)] = [(\frac{1}{\beta} - 1)(\sigma-1)]^{1/\sigma}[1 - \beta(1-\delta) - \xi(1-\beta)] \tag{540}$$

Plug combining last three equations:

$$1 > \left[(1-\delta) + (R + Q\xi)\,\frac{\sigma}{\sigma-1}Q^{\sigma-1}\right]$$

we solve the range of ξ.

14.3. *** Plantin (2021).** This paper introduces a "monetary bubble" as unintended consequence of monetary easing, which is distinguished from "natural bubbles" arising from low interest rate in three aspects: (1) monetary bubble is stochastic and must burst when monetary easing ends; (2) monetary bubble doesn't raise interest rate controlled by monetary authority; (3) monetary bubble always crowds out investment and reduce utility of productive, constrained entrepreneurs.

Set Up. The economy is discrete in time and populated by household, entrepreneur and a monetary authority.

- Household: live for two period
  - young: risk averse, supply labor
  - old: risk neutral
  - maximize utility
  $$u(C_Y) + \beta C_O - \gamma \frac{L^2}{2}$$
  → labor supply
  $$wu'(C_Y) = \gamma L$$
  → euler
  $$u'(C_Y) = \beta r_t$$
  → saving
  $$C_O = w_t L_t - C_Y$$
  - consumption bundle:
  $$C_t = \left(\int_0^1 C_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}}$$
- Entrepreneur: live for two period,
  - young: endowed with *production tech + investment tech*
    - production tech: L unit of labor → αL unit of interm' good i, $\alpha > 0$:
    $$\max_{P_i} P_i Y_i - \frac{W_t Y_i}{\alpha}$$
    → real wage
    $$w_t = \frac{\alpha(\varepsilon - 1)}{\varepsilon}$$
    → mark-up
    $$\mu_t = \alpha - w = \frac{\alpha}{\varepsilon}$$
    → profit: $\mu_t L_t$
    - investment tech: save $x_t \to c_{t+1} = \rho x_t$, $\rho > 1$:
    $$\max_{a,I,R_E} R_E + r_t(\mu L_t - a)$$
    s.t. IR constraint of HH ($I - a$: investment by HH, R_E: return to entrepreneur)
    $$\rho I - R_E \geq r_t(I - a)$$
    IC constraint from financial friction
    $$R_E \geq (1 - \lambda)\rho I$$

solution: if $\rho < r_t$:

$$I = a = R_E = 0$$

solution: if $\rho > r_t > \lambda\rho$:[58]

$$I = \frac{\mu L_t r_t}{r_t - \lambda\rho}; \quad a = \mu L_t; \quad R_E = (1-\alpha)\rho I$$

solution: if $\lambda\rho \geq r_t$:

$$I = +\infty$$

solution: if $\rho = r_t$:

$$I \in \left(0, \frac{\mu L_t r_t}{r_t - \lambda\rho}\right); \quad a = \mu L_t; \quad R_E = \rho a \geq (1-\alpha)\rho I$$

- old: consume

- Monetary authority: gross nominal interest rate R_t
  - interest rate rule:

$$R_t = r_t \left(\frac{P_t}{P_{t-1}}\right)^{1+\psi}$$

  - r_t: real interest rate consistent with household problem:

$$u'(C_Y) = \beta r$$

Non-Bubble Perfect Foresight Equilibrium. The model admits a unique non-bubble equilibrium. We now solve it.
Saving. Household's saving is given by

$$s^H(r) \equiv w * L - C_Y = \frac{\alpha(\varepsilon-1)}{\varepsilon} * \frac{\frac{\alpha(\varepsilon-1)}{\varepsilon}\beta r}{\gamma} - (u')^{-1}(\beta r)$$

Entrepreneur's saving is given by

$$s^E(r) \equiv (\alpha - w)L = (\alpha - \frac{\alpha(\varepsilon-1)}{\varepsilon}) * \frac{\frac{\alpha(\varepsilon-1)}{\varepsilon}\beta r}{\gamma}$$

Aggregate saving thus becomes

$$s(r) \equiv s^H(r) + s^E(r) = \frac{\alpha^2(\varepsilon-1)\beta r}{\varepsilon\gamma} - (u')^{-1}(\beta r)$$

Investment.

- If $r_t > \rho$:

$$I = 0$$

- If $r \leq \lambda\rho$:

$$I = \infty$$

- If $r \in (\lambda\rho, \rho)$:

$$I(r) = \frac{\mu L r}{r - \lambda\rho} = \frac{\frac{\alpha^2(\varepsilon-1)\beta}{\varepsilon^2\gamma} r^2}{r - \lambda\rho} = \underbrace{\frac{\alpha^2(\varepsilon-1)\beta}{\varepsilon^2\gamma} r}_{\text{net-worth}} \underbrace{\frac{1}{1-\lambda\rho/r}}_{\text{leverage}}$$

  note that $I'(r) = 0 \to r = 2\lambda\rho$, so there are two scenarios:

[58] There is a type at Appendix A.1, pp. 20. in the paper.

– $2\lambda\rho \geq \rho$ ($\lambda \geq 0.05$): $I(r)$ is strictly decreasing in r (leverage effect dominates).
– $2\lambda\rho < \rho$ ($\lambda < 0.05$): $I(r)$ decreases then increases in r (net-worth effect dominates).

Equilibrium. Saving equals investment in equilibrium. Figure 35 shows existence of unique equilibrium.

$$I(r) \equiv \frac{\frac{\alpha^2(\varepsilon-1)\beta}{\varepsilon^2\gamma} r^2}{r - \lambda\rho} = \frac{\alpha^2(\varepsilon-1)\beta r}{\varepsilon\gamma} - (u')^{-1}(\beta r) \equiv S(r)$$

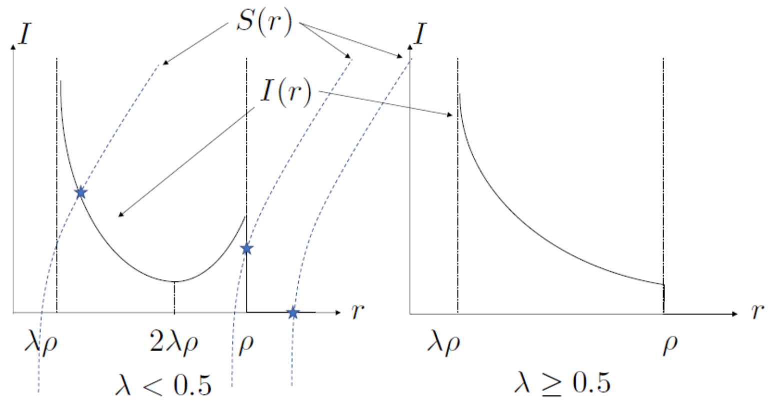


FIGURE 35. Non-bubbly Equilibrium

Natural Bubbly Equilibrium. The model can admit a bubbly equilibrium, i.e. a constant-size bubble is refinanced at the unit interest rate, when return on external finance is low $r < 1 < \rho$.

Given that $r < \rho$, investment is given by

$$I(r) \equiv \frac{\frac{\alpha^2(\varepsilon-1)\beta}{\varepsilon^2\gamma} r^2}{r - \lambda\rho} = \frac{\alpha^2(\varepsilon-1)\beta r}{\varepsilon\gamma} - (u')^{-1}(\beta r) \equiv S(r)$$

If a bubble can be refinanced at $r = 1$, the bubbly equilibrium features

$$I(r) \equiv \frac{\frac{\alpha^2(\varepsilon-1)\beta}{\varepsilon^2\gamma}}{1 - \lambda\rho} = \frac{\alpha^2(\varepsilon-1)\beta}{\varepsilon\gamma} - (u')^{-1}(\beta) - B \equiv S(r) - B$$

In other word, the bubble fills in the wedge between aggregate saving and investment. Thus bubble raises interest rate r, and (see Figure 36)

- if $\lambda \geq 0.5$, bubble always crowds out investment (leverage effect dominates)
- if $\lambda < 0.5$, bubble may crowd in investment when $r > r_I$ (i.e. at E') when net-worth effect dominates; bubble may crowd out investment when $r < r_I$ (i.e. at E).

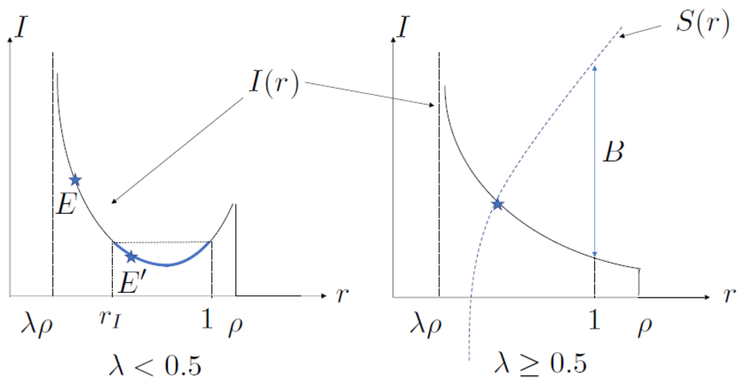


FIGURE 36. Natural Bubbly Equilibrium

Monetary Policy and Monetary Bubble. We start with a proposition on equivalence between real interest rate r and elasticity of substitution ε: For $r \in (\underline{r}, \rho)$, there is a unique (decreasing) one-to-one mapping between ε and r, denoted as $\varepsilon = \eta(r)$. This implies that a monetary policy temporarily controlling real interest rate acts like controlling ε.

Now we introduce uncertainty into the model, i.e. financial crisis (low λ) ends stochastically and followed by permanent normal times.

- crisis state: $\lambda_t = \lambda_c$; corresponding to natural rate $= r_c$
  - w.p. p: (stay in crisis) $\lambda_{t+1} = \lambda_c$
  - w.p. $1 - p$: (back to normal) $\lambda_{t+1} = \lambda_n$, such that $\lambda_n > \lambda_c$
  - monetary policy at crisis state:

$$R_t = \hat{r} \in (\underline{r}, \rho)$$

- normal state: $\lambda_t = \lambda_n$; corresponding to natural rate $= r_n$
  - monetary policy at normal state: Taylor Rule

$$R_t = r_t \left(\frac{P_t}{P_{t-1}} \right)^{1+\psi}$$

Non-bubbly equilibrium.

- price must be constant in non-bubbly equilibrium.
  - (nominal rigidity) price are set one period in advance (*before knowing crisis ends*)
  - (commitment to Taylor rule post-crisis) $P_t = P_{t-1}$ once crisis ends
  - $\rightarrow$ price must be constant ($= P_{-1}$) in crisis
- real interest rate r_t equals policy rate $\hat{r}_t$
  - $\hat{r}_t$ set in accordance with actual elasticity of substitution: $\hat{r}_t = \eta^{-1}(\varepsilon)$
  - full control over real interest rate in crisis

Characterization of non-bubbly equilibrium

- implied elasticity $\eta(\hat{r}) = \varepsilon$ as in non-bubbly equilibrium without crisis

$$I(\hat{r}) \equiv \frac{\frac{\alpha^2(\eta(\hat{r})-1)\beta}{\eta(\hat{r})^2\gamma}\hat{r}^2}{\hat{r} - \lambda\rho} = \frac{\alpha^2(\eta(\hat{r}) - 1)\beta\hat{r}}{\eta(\hat{r})\gamma} - (u')^{-1}(\beta\hat{r}) \equiv S(\hat{r})$$

- output:

$$y = \alpha L = \frac{\alpha^2(\eta(\hat{r})-1)\beta\eta(\hat{r})}{\eta(\hat{r})\gamma} = \frac{\alpha^2\beta\rho\lambda}{2\gamma}\left(1+\sqrt{1+\frac{4\gamma\phi(\beta\hat{r})(\hat{r}-\rho\lambda)}{\alpha^2\beta^2\rho^2\lambda^2}}\right)$$

- investment:

$$I = y - \phi(\beta\hat{r})$$

- assumption: $1 < r_c < r_n < \rho$
  - $\rightarrow r_c > 1$: no *natural* bubble
  - $\rightarrow$ if $\hat{r} = r_c$ (neutral MP): no bubble

Monetary bubbly equilibrium. If monetary bubble exists, it must satisfies:

- bubble must burst when crisis ends (w.p. $1-p$) $\rightarrow$ stochastic bubble
- bubble must grow at $\hat{r}$ (controlled by monetary authority)
- $\rightarrow$ bubble can only exist during crisis and only if $\hat{r} \leq p$
- $\rightarrow$ monetary policy are bubble-proof when $\hat{r} > p$

We construct such *stochastic* monetary bubble (b_t) such that (ε_t^b denote *shadow* elasticity of substitution)

$$I(\hat{r}) \equiv \frac{\frac{\alpha^2(\varepsilon_0^b-1)\beta}{\varepsilon_0^{b2}\gamma}\hat{r}^2}{\hat{r}-\lambda\rho} = \frac{\alpha^2(\varepsilon_0^b-1)\beta\hat{r}}{\varepsilon_0^b\gamma} - (u')^{-1}(\beta\hat{r}) - b_0 \equiv S(\hat{r}) - b_t, \quad t=0$$

$$I(\hat{r}) \equiv \frac{\frac{\alpha^2(\varepsilon_t^b-1)\beta}{\varepsilon_t^{b2}\gamma}\hat{r}^2}{\hat{r}-\lambda\rho} = \frac{\alpha^2(\varepsilon_t^b-1)\beta\hat{r}}{\varepsilon_t^b\gamma} - (u')^{-1}(\beta\hat{r}) - \frac{\hat{r}}{p}b_{t-1} \equiv S(\hat{r}) - b_t, \quad t>0$$

Effects of *monetary bubble* (vs. effects of natural bubble):

- expected return on asset is unchanged: $\hat{r}$ (vs. natural bubbles raise return)
- holding $\hat{r}$ constant, shadow ε_t^b must be higher in bubbly equilibrium.
- output is higher: as *shadow* $\varepsilon_t^b > \eta(\hat{r})$ (vs. higher or lower)
- investment is crowded out: $I\downarrow$ (vs. crowd in or out)
- entrepreneur's utility is lower: $I\downarrow \rightarrow (\rho-\hat{r})I$ (vs. higher or lower)
- labor share is higher: $\varepsilon_t^b \uparrow \rightarrow \frac{wL}{\alpha L}\uparrow$
- household's utility is higher: as $wL\uparrow$

15. Search and Matching in Macroeconomics (Housing)

15.1. ***Genesove and Han (2012).** Genesove and Han (2012) introduces a *basic random matching model of housing.*

Set-up. Denote

- number of risk-neutral buyers and sellers: n_b and n_s.
- market thickness: $\theta = \frac{n_b}{n_s}$.
- contact (meeting) rate: $m(n_b, n_s)$ constant return.
- contact probability of seller:

$$q_s(\theta) = m/n_s = m(\theta, 1) \equiv \theta q_b(\theta), \quad \text{where}$$

- contact probability of buyer:

$$q_b(\theta) = m/n_b = m(1, 1/\theta)$$

- all buyer–seller pairs are ex ante identical
- idiosyncratic match utility (known to buyer and seller) of a house for a buyer: $x_{ij} \sim g(x)$.
- reservation of buyer and seller (V_b, V_s): $x^* \equiv V_b^* + V_s^*$
- total surplus from a match: $x - x^*$
- probability of transaction conditional on a meeting: $1 - G(x^*) = prob(x \geq x^*)$
- expected surplus conditional on a transaction: $E(x|x - x^* \geq 0) - x^*$
- search cost for buyers and sellers: (c_b, c_s).

Assumption.

- V_b^* is exogenous: many markets for buyer to choose from (an infinite supply of buyers)
- Nash bargaining b/w seller and buyer over potential surplus: β to buyer; $1 - \beta$ to seller

Asset pricing equations. Given a transaction, expected price:

$$p = V_s^* + (1 - \beta)[E(x - x^*|x - x^* \geq 0)]$$

Optimal conditions for buyers and sellers:

$$\underbrace{rV_s^* + c_s}_{\text{cost}} = \underbrace{q_s(\theta)}_{\text{contact rate}} \underbrace{[1 - G(x^*)]}_{\text{acceptance rate}} \underbrace{(1 - \beta)[E(x - x^*|x - x^* \geq 0)]}_{\textit{seller's payoff in trans.}} \tag{541}$$

[59] and

$$rV_b^* + c_b = q_b(\theta)[1 - G(x^*)]\beta[E(x - x^*|x - x^* \geq 0)] \tag{542}$$

We have two equations and two unknowns (θ and V_s^*) given exogenous V_b^*

[59]While there is a benefit to continued search arising from the potential of a more beneficial future match, search is also assumed to entail costs. These costs consist of both direct search costs, and indirect costs, because agents discount the future at the rate r. Due to these costs agents prefer, ceteris paribus, to transact sooner rather than later.

Properties of basic model. Assume that acceptance rate $1 - G(x) = \Phi(x - \nu)$, where ν is a parameter governing housing demand. When ν increases,

- housing demand (the expected surplus from a transaction) $\uparrow$
- demand curve (x^* on y-xis and θ on x-axis) $\uparrow$ one for one but supply curve $\uparrow$ less than one for one [60]
- equilibrium θ and x^* $\uparrow$, but $(x^* - \nu)$ $\downarrow$
- $(x^* - \nu)$ $\downarrow$: the acceptance rate $1 - G(x^*)$ $\uparrow$
- θ $\uparrow$: contact rate $q_s(\theta)$ $\uparrow$
- $\Rightarrow$ probability of selling $q_s(\theta)[1 - G(x^*)]$ $\uparrow$
- $\Rightarrow$ seller time on the market (STOM) $\downarrow$
- $\Rightarrow$ probability of purchase $q_b(\theta)[1 - G(x^*)]$ ambiguous

Summary on effect of demand shock: market tightness $\uparrow$ $\rightarrow$ seller contact hazard $\uparrow$ & buyer contact hazard $\downarrow$ $\Rightarrow$ probability of selling $\uparrow$ $\rightarrow$ STOM $\downarrow$

[60] proof by contradiction: Suppose that surve curve change one-for-one so that θ is unchanged. Expected surplus (RHS of Eq. (542)) is unchanged since V_b^* is constant. Assume supply curve shifts up one-for-one, then $\Phi(x^* - \nu)$ or $x^* - \nu$ must be unchanged. x^* must increase as ν increases. But from Eq. (541), V_s^* must not be changed.

15.2. *** Novy-Marx (2009).** Novy-Marx (2009) relaxes the assumption of exogenous buyer's utility in the basic random matching model and assumes that both buyer entry and seller entry depend on the payoffs of buying and selling a house. This generates *endogenous entry* that amplifies the shock.

Assumptions. Using the notation from last section (Genesove and Han, 2012), we introduce two new assumptions on

- (for concreteness) matching fun: $q_b(\theta) = \lambda\theta^\eta$, where $\lambda = q(1)$.
- (key assumption) endogenous entry (but in reduced-form):
$$F_i = X_i(V_i + \frac{c_i}{r})^\gamma,$$
where V_i denotes expected value of entry; F_i denotes the entry rate; γ denotes elasticity of entry to expected payoff; the term $\frac{c_i}{r}$ is added for technical purpose.

Pricing equations. The continuation value of walking away from a potential match to agent i (buyer or seller) is

$$V_i^* = \mathbf{E}\left[e^{-r\tau_i}\right]\left(G(x^*)V_i^* + (1 - G(x^*))\,\mathbf{E}\left[V_i \mid x > x^*\right]\right) - \int_0^{\tau_i} c_i e^{-rt} dt$$

where τ_i is the time until the agent meets the next potential counter-party. Under Nash bargaining which maximizes $(V_b - V_b^*)^\beta (V_s - V_s^*)^{1-\beta}$, we have

$$\mathbf{E}\left[V_i \mid x > x^*\right] = V_i^* + \beta_i \mathbf{E}\left[x - x^* \mid x > x^*\right]$$

Recall the encounter rate for type i agents (i.e., $q_b(\theta) = \theta^\eta\lambda$ and $q_s(\theta) = \theta\lambda_b$), we have

$$\mathbf{E}\left[e^{-r\tau_i}\right] = \int_0^\infty e^{-rt} q_i(\theta) e^{-q_i(\theta)t} dt = \frac{q_i(\theta)}{r + q_i(\theta)},$$

$$\int_0^{\tau_i} c_i e^{-rt} dt = c_i \int_0^\infty \left(\int_0^t e^{-rs} ds\right) q_i(\theta) e^{-q_i(\theta)t} dt = \frac{c_i}{r + q_i(\theta)}.$$

Together, we obtain two equations parallel to eq. (541) and eq. (542)

$$\underbrace{rV_i^* + c_i}_{\text{cost of continued search}} = \underbrace{q_i(\theta)\beta_i \mathbf{E}\left[x - x^* \mid x > x^*\right]}_{\text{expected gain}} \tag{543}$$

Combining eq.(543) for buyers and sellers, we obtain:

$$r(x^*) + c_s + c_b = \lambda[\beta\theta^\eta + (1 - \beta)\theta^{1+\eta}]\mathbf{E}\left[x - x^* \mid x > x^*\right]$$

which (uniquely) solves x^* given $\lambda, \theta, \eta, c_i$ and r.

Time-to-Transaction. Now we derive expected time agents search prior to successfully transacting, denoted as $E[T_i]$ as a function of market tightness, θ. Because of the Markovian nature of the market,

$$\mathbf{E}\left[T_i\right] = \mathbf{E}\left[\tau_i\right] + \mathbf{P}\left[x < x^*\right]\mathbf{E}\left[T_i\right]$$

Rearranging the previous equation yields

$$\mathbf{E}\left[T_i\right] = \frac{\mathbf{E}\left[\tau_i\right]}{\mathbf{P}\left[x \geq x^*\right]},$$

which, using $\mathbf{E}\left[\tau_i\right] = [q_i(\theta)]^{-1}$ and $\mathbf{P}\left[x < x^*\right] = G\left(x^*\right)$, implies

$$\mathbf{E}\left[T_b\right] = \frac{\theta^{-\eta}}{\left(1 - G\left(x^*\right)\right)\lambda}$$

Figure 2 ■ Threat-points and required transaction utility.

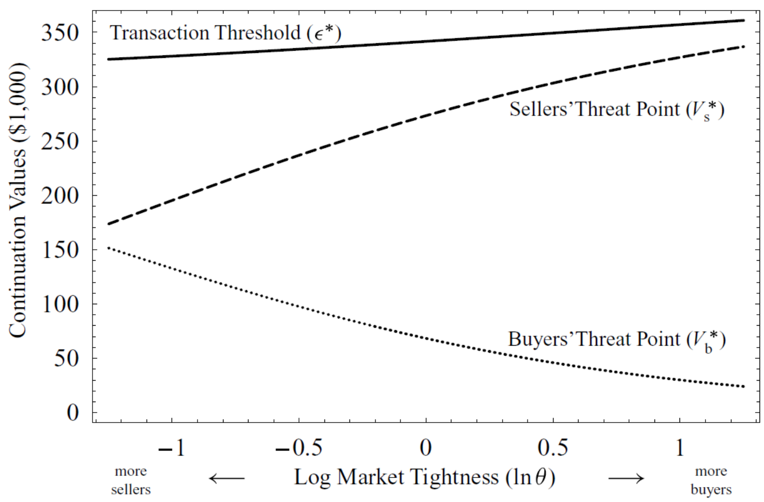
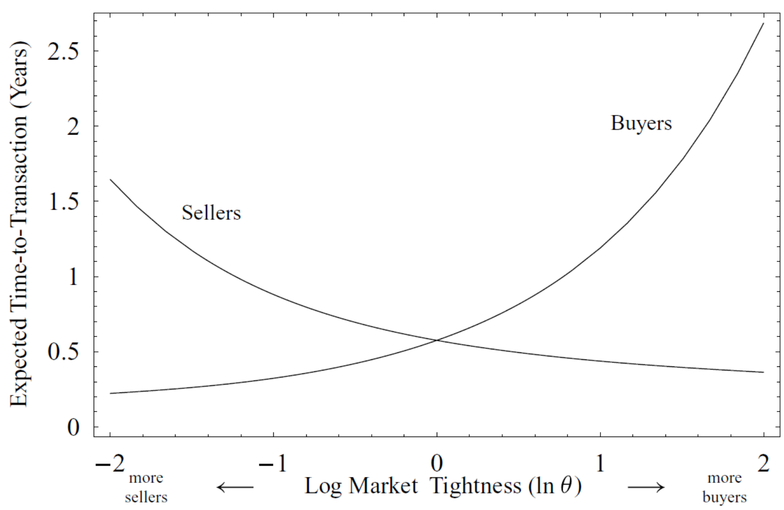


Figure 3 ■ Expected time-to-transaction.



and $\mathbf{E}[T_s] = \mathbf{E}[T_b]/\theta$.

Stationary Equilibrium. Now we incorporate the mechanism into a macroeconomic framework and consider how (1) entry decision of potential market participants, (2) the exit decision of current participants, and (3) the behavior of those agents actively searching in the market interact.

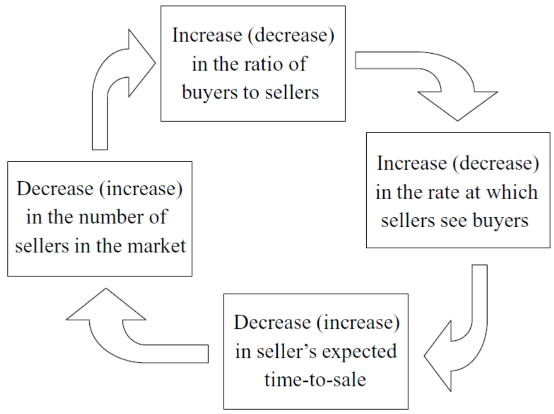
In a stationary equilibrium w. entry and exit,

- buyers and sellers both enter the market at the same rate they exit;
- buyers and sellers exit when they transact;
- buyers and sellers exit the market at the same rate

which implies that buyers and sellers enter the market at the same rate such that $F_b(V_b^*) = F_s(V_s^*)$, or or

$$X_b(V_b^* + \frac{c_b}{r})^\gamma = X_s(V_s^* + \frac{c_s}{r})^\gamma$$

Figure 1 ■ Feedback in search markets.



Last equation, combined with optimal search conditions of buyers and sellers:

$$rV_i^* + c_i = q_i(\theta)\beta_i \mathbf{E}\left[x - x^* \mid x > x^*\right]$$

delivers

$$X_b(\frac{q_b(\theta)\beta_b \mathbf{E}\left[x - x^* \mid x > x^*\right]}{r})^\gamma = X_s(\frac{q_s(\theta)(1-\beta)\mathbf{E}\left[x - x^* \mid x > x^*\right]}{r})^\gamma$$

or equivalently,

$$\underbrace{\log\left[\theta\right]}_{\text{market thickness}} = \frac{1}{\gamma} \underbrace{\log\left[(\frac{1-\beta}{\beta})^\gamma \frac{X_b}{X_s}\right]}_{\text{buyers' relative propensity to enter}}$$

Last equation says a 1% shift in the relative level of the supply curves for new entrants of buyers and sellers results in a $1/\gamma$ percent change in the ratio of buyers to sellers searching at any given time (*versus 0 percent in standard search literature*).

Amplification w. endogenous entry. Shocks to supply or demand ⇒ shocks to market tightness ⇒ greater impact on prices and expected search times.

15.3. *** Piazzesi and Schneider (2009).** Piazzesi and Schneider (2009) incorporates *behaviour components* into a search model and shows how a group of irrational (optimistic) buyers can generate momentum in the housing market.

References

Basu, S. and B. Bundick (2017). Uncertainty shocks in a model of effective demand. *Econometrica 85*(3), 937–958.

Benigno, G., H. Chen, C. Otrok, A. Rebucci, and E. R. Young (2013). Financial crises and macro-prudential policies. *Journal of International Economics 89*(2), 453–470.

Bernanke, B. S., M. Gertler, and S. Gilchrist (1999). The financial accelerator in a quantitative business cycle framework. *Handbook of Macroeconomics 1*, 1341–1393.

Bils, M., P. J. Klenow, and C. Ruane (2017). Misallocation or mismeasurement? *Society for Economic Dynamics*.

Bloom, N., M. Floetotto, N. Jaimovich, I. Saporta-Eksten, and S. J. Terry (2018). Really uncertain business cycles. *Econometrica 86*(3), 1031–1065.

Brunnermeier, M. K. (2009). Deciphering the liquidity and credit crunch 2007-2008. *Journal of Economic perspectives 23*(1), 77–100.

David, J. M. and V. Venkateswaran (2019). The sources of capital misallocation. *American Economic Review 109*(7), 2531–67.

Eisfeldt, A. L. and D. Papanikolaou (2013). Organization capital and the cross-section of expected returns. *The Journal of Finance 68*(4), 1365–1406.

Fama, E. F. and K. R. French (1992). The cross-section of expected stock returns. *the Journal of Finance 47*(2), 427–465.

Fernández-Villaverde, J., P. Guerrón-Quintana, J. F. Rubio-Ramírez, and M. Uribe (2011). Risk matters: The real effects of volatility shocks. *American Economic Review 101*(6), 2530–61.

Fishman, M. J., J. A. Parker, L. Straub, et al. (2019). A dynamic theory of lending standards. In *2019 Meeting Papers*, Number 1344. Society for Economic Dynamics.

Fostel, A. and J. Geanakoplos (2014). Endogenous collateral constraints and the leverage cycle. *Annu. Rev. Econ. 6*(1), 771–799.

Geanakoplos, J. (2010). The leverage cycle. *NBER macroeconomics annual 24*(1), 1–66.

Genesove, D. and L. Han (2012). Search and matching in the housing market. *Journal of Urban Economics 72*(1), 31–45.

Gilchrist, S. and E. Zakrajšek (2012). Credit spreads and business cycle fluctuations. *American Economic Review 102*(4), 1692–1720.

Hellwig, M. (1987). Some recent developments in the theory of competition in markets with adverse selection. *European Economic Review 31*(1-2), 319–325.

Hu, Y. (2017). A dynamic theory of bank lending, firm entry, and investment fluctuations. *Firm Entry, and Investment Fluctuations (August 11, 2017)*.

Khan, A. and J. K. Thomas (2008). Idiosyncratic shocks and the role of nonconvexities in plant and aggregate investment dynamics. *Econometrica 76*(2), 395–436.

Korinek, A. (2018). Regulating capital flows to emerging markets: An externality view. *Journal of International Economics 111*, 61–80.

Leduc, S. and Z. Liu (2016). Uncertainty shocks are aggregate demand shocks. *Journal of Monetary Economics 82*, 20–35.

Li, E. X., L. X. Liu, and C. Xue (2014). Intangible assets and cross-sectional stock returns: Evidence from structural estimation. *Available at SSRN 1560014*.

Miao, J. and P. Wang (2018). Asset bubbles and credit constraints. *American Economic Review 108*(9), 2590–2628.

Nenov, P. T. (2016). Endogenous leverage and advantageous selection in credit markets. *The Review of Financial Studies 30*(11), 3888–3920.

Novy-Marx, R. (2009). Hot and cold markets. *Real Estate Economics 37*(1), 1–22.

Peters, R. H. and L. A. Taylor (2017). Intangible capital and the investment-q relation. *Journal of Financial Economics 123*(2), 251–272.

Piazzesi, M. and M. Schneider (2009). Momentum traders in the housing market: Survey evidence and a search model. *American Economic Review 99*(2), 406–11.

Shiller, R. J. (2005). *Irrational Exuberance Ed. 2.* Princeton University Press.

Taddei, F. (2010). Liquidity and economic fluctuations. Technical report, Collegio Carlo Alberto.