Problem Set 2: Solution

Computation Study Group

June 2021

Task 2: Uncertainty Shock w. Flexible Price

In this task we solve a simplified model of Dong, Liu and Wang (2021) with flexible price. The model features heterogeneous firms and endogenous misallocation.

Setting. Consider an economy with a continuum of firms that produce with a linear technology using labor n_{jt} as single input:

$$y_{jt} = A_t z_{jt} n_{jt} \tag{1}$$

where A_t measures aggregate productivity, and z_{jt} measures idiosyncratic productivity.

The process of idiosyncratic productivity is assumed to follow the following process:

$$z_{jt+1} = \begin{cases} z_{jt} & w.p. \quad \rho_t \\ \tilde{z} & w.p. \quad 1-\rho_t \end{cases} \tag{2}$$

where $\tilde{z}$ is discrete random variable with $\tilde{z} = z_j$ occurring with probability π_j, $j = 1, 2, ..., I$. We assume that $z_1 < z_2 < ... < z_I$ without loss of generality. The process features time-invariant cross-sectional distribution of firm productivity such that, regardless of realization on ρ_t, there are always π_j fraction of firms with $z_{jt} = z_j$ in each period.

The firms' problem is given by the following Bellman equation (we suppress aggregate state in notation for simple exposition):

$$V_t(z_{jt}, \tau_{jt}) = \tau_{jt} A_t z_{jt} n_{jt} - W_t n_{jt} + \beta E_t V_{t+1}(z_{jt+1}, \tau_{jt+1}) \tag{3}$$

subject to a credit constraint

$$W_t n_{jt} \leq \theta_t \beta E_t V_{t+1}(z_{jt+1}, \tau_{jt+1}) \tag{4}$$

where θ_t is a financial shock measuring tightness of credit constraint. τ_{jt} is idiosyncratic distortion (net subsidy) on output, and is assumed to be an i.i.d. random variable with cumulative distribution function $F(\tau)$. Denote

$$\bar{V}_{jt} = \int V_t(z_{jt}, \tau_{jt}) dF(\tau), \tag{5}$$

we can write discounted future value conditional on current realization of productivity as

$$\beta E_t V_{t+1}(z_{jt+1}, \tau_{jt+1} | z_{jt} = z_j) = \beta E_t \left[\rho_t \bar{V}_{jt+1} + (1-\rho_t) \sum_{i=1}^{I} \pi_i \bar{V}_{it+1} \right] \equiv B_{jt} \tag{6}$$

Solving static profit maximization problem gives the allocations of production and credit: a firm will borrow and produce in current period if and only if net subsidy τ_{jt} is higher than a cut-off:

$$\hat{\tau}_{jt} \equiv \hat{\tau}_t(z_{jt}) = \frac{W_t}{A_t z_{jt}} \tag{7}$$

Without loss of generality we assume marginal firms operate. Firms with relatively higher productivity (z_{jt}) and subsidy (or lower tax) choose to produce and borrow up to the limit to finance wage bill. Low productivity or heavily taxed firms stay inactive and do not borrow. Therefore, idiosyncratic labor demand function is

$$n_t(z_{jt}, \tau_{jt}) = \begin{cases} \frac{\theta_t B_{jt}}{W_t}, & \textit{if} \quad \tau_{jt} \geq \hat{\tau}_{jt} \\ 0, & \textit{otherwise} \end{cases} \tag{8}$$

The value function in equation (3) can be re-written as

$$V_t(z_{jt}, \tau_{jt}) = \max(\frac{A_t z_{jt} \tau_{jt}}{W_t} - 1, 0)\theta_t B_{jt} + B_{jt} \tag{9}$$

Define aggregate firm value as

$$\bar{V}_t = \sum_{i=1}^{I} \pi_i \bar{V}_{it} \tag{10}$$

It follows that for $j = 1, 2, ..., I$,

$$\bar{V}_{jt} = \left[1 + \theta_t \int_{\frac{W_t}{A_t z_{jt}}} \left(\frac{A_t z_{jt}}{W_t}\tau - 1\right) dF(\tau)\right] \beta E_t \left[\rho_t \bar{V}_{jt+1} + (1-\rho_t)\bar{V}_t\right] \equiv \Phi(\frac{W_t}{A_t z_{jt}}, \theta_t) B_{jt} \tag{11}$$

where $\Phi(\frac{W_t}{A_t z_{jt}}, \theta_t) \equiv 1 + \theta_t \int_{\frac{W_t}{A_t z_{jt}}} \left(\frac{A_t z_{jt}}{W_t}\tau - 1\right) dF(\tau)$. It is clear that Φ is an increasing function of z_{jt}, θ_t, and decreasing function of W_t.
To solve the equilibrium, we need to impose a labor market clearing condition:

$$N_t = \sum_{j=1}^{I} \pi_j \int_{\tau} n_t(z_{jt}, \tau) dF(\tau) \equiv \sum_{j=1}^{I} \pi_j N_{jt}, \tag{12}$$

where N_t is exogenous labor supply and N_{jt} for $j = 1, 2, ..., I$ is

$$N_{jt} = \frac{\theta_t \beta E_t \left[\rho_t \bar{V}_{jt+1} + (1-\rho_t)\bar{V}_t\right]}{W_t} \left[1 - F(\frac{W_t}{A_t z_{jt}})\right] \tag{13}$$

The aggregate output (Y_t) is given by

$$Y_t = \sum_{j=1}^{I} \pi_j \frac{A_t z_{jt} \theta_t B_{jt}}{W_t} [1 - F(\frac{W_t}{A_t z_{jt}})] = A_t \sum_{j=1}^{I} \pi_j z_{jt} N_{jt} \tag{14}$$

and endogenous TFP, denoted as Z_t, is defined as

$$Z_t \equiv \frac{Y_t}{A_t N_t} = \frac{\sum_{j=1}^{I} \pi_j z_{jt} N_{jt}}{N_t} \tag{15}$$

Last equation shows that, given exogenous labor supply, endogenous TFP reflects labor misallocation.

Steady State We now solve the steady state. Let us define $\sum_{i=1}^{I} \pi_i \bar{V}_i = \bar{V}$. Then equation (10) at steady state implies that

$$\bar{V}_j = \frac{\beta(1-\rho)\Phi(\frac{W}{z_j}, \theta)}{1-\beta\rho\Phi(\frac{W}{z_j}, \theta)} \bar{V}$$

Then by definition

$$\sum_{j=1}^{I} \pi_j \bar{V}_j \equiv \sum_{j=1}^{I} \pi_j \frac{\beta(1-\rho)\Phi(\frac{W}{z_j}, \theta)}{1-\beta\rho\Phi(\frac{W}{z_j}, \theta)} \bar{V} = \bar{V}$$

which implies that

$$G(W, \rho, \theta) \equiv \sum_{j=1}^{I} \pi_j \frac{\beta(1-\rho)\Phi(\frac{W}{z_j}, \theta)}{1-\beta\rho\Phi(\frac{W}{z_j}, \theta)} = 1 \tag{16}$$

Last equation solves the steady state wage W. With W solved, the steady state of other variables are straightforward to derive.

Calibration. Assume that τ follows a log-normal distribution

$$log(\tau) \sim N(\mu_t, \sigma_t), \quad \mu_t = -0.5\sigma_t^2$$

Set $\beta = 0.96$; $\sigma_{ss} = 0.80$; $\theta = 0.35$; $\rho_{ss} = 0.82$; Labor supply is 0.33. z_j is discretized into 10 states from an AR(1) process, with $\rho_z = 0.80$ and $\sigma_z = 0.20$, using Tauchen method.

Task. Write down the competitive equilibrium is defined as a sequence of variables

$$\{N_{jt}, \bar{V}_{jt}, \bar{V}_t, W_t, Y_t, Z_t\}$$

uniquely determined by equation (10) - (15), given exogenous processes of exogenous variables: $\{\rho_t, A_t, \theta_t\}$.

(1) Assume ρ_t follows the following process:

$$\log(\rho_t) = (1-\rho^\rho)\log(\bar{\rho}) + \rho^\rho \log(\rho_{t-1}) - \sigma^\rho \varepsilon_t^\rho, \quad \varepsilon_t^\rho \sim N(0,1) \tag{17}$$

where $\bar{\rho} = 0.82$, $\rho^\rho = 0.84$, $\sigma^\rho = 0.10$. Simulate the economy against this ρ_t shock.

(2) Assume σ_t follows the following process:

$$\log(\sigma_t) = (1-\rho^\sigma)\log(\bar{\sigma}) + \rho^\sigma \log(\sigma_{t-1}) - \sigma^\sigma \varepsilon_t^\sigma, \quad \varepsilon_t^\sigma \sim N(0,1) \tag{18}$$

where $\bar{\sigma} = 0.80$, $\rho^\rho = 0.70$, $\sigma^\rho = 0.01$. Simulate the economy against this σ_t shock.

Sol. See attached codes.