

# Firm Dynamics I

Hopenhayn (1992)  
Cooley and Quadrini (2001)

Prepared By Ding Dong  
Department of Economics, HKUST

# Stylized Facts about Firm Dynamics

- Size
  - size distribution of firms is skewed to the right and
  - the skewness of a cohorts size distribution declines with age
- Investment
  - investment growth decreases with size and age, both unconditionally and conditionally
- Employment
  - employment growth decreases with size and age, both unconditionally and conditionally
- Entry and exit
  - exit hazard rate declines with age
  - entry rate is procyclical
  - exit rate is countercyclical

# Classic Models of Firm Dynamics

- \*Hopenhayn (ECMA, 1992): workhorse model
  - endogenous stationary distribution with entry-and-exit
  - no aggregate uncertainty
  - frictionless (except a fixed operation cost)
- Gomes (AER, 2001)
  - capital accumulation under financial frictions
- \*Cooley and Quadrini (AER, 2001)
  - financial frictions
  - persistent idiosyncratic shock
- Clementi and Palazzo (AEJ: Macro, 2016)
  - aggregate shock with real friction
- Begenau and Salomao (RFS, 2018): external financing cycles
  - aggregate shock with financial friction

# Hopenhayn (1992)

- Discrete and infinite time horizon
  - discount factor:  $\beta$
- Continuum of firms
  - law of large numbers holds
- Homogeneous product with single input
  - single input: labor
  - exogenous aggregate demand for output
  - exogenous aggregate supply of input
- Entry and exit
  - potential entrants are *ex ante* identical
  - incumbents are heterogeneous in idiosyncratic productivity

# Firm's Problem



- Production technology:

$$f(a, n) = ay(n) = an^\alpha$$

- a: idiosyncratic productivity, Markov process:  $a \rightarrow a'$
- labor input
- $\alpha < 1$ : decreasing return to scale  $\rightarrow$  optimal size
- Role of fixed cost:  $c^f$ 
  - generating endogenous exit
- Operating profit:

$$\pi(a, p, w) = \max_n pf(a, n) - wn - c^f$$

- optimal output denoted as  $q^* := f(a, n^*)$
- optimal input denoted as  $n^* := n(a, p, w)$

# Incumbent's Problem

- Two decisions:
  - size of employment: one-to-one mapping from productivity ( $a$ )
  - exit
- Exit decision:
  - if exit: 0
  - if not exit: *expected* operating profit
- Value function:

$$v_t(a; \mu) = \pi(a, p, w) + \beta \max\{0, \int v_{t+1}(a'; \mu') F(da'|a)\}$$

- $\mu$  : aggregate state (i.e., distribution, thus prices)
- exit cut-off value  $a^*$ :

$$0 = \int v_{t+1}(a'; \mu') F(da'|a^*) \quad \text{or} \\ a^* = \inf\{a \in A : \int v_{t+1}(a'; \mu') F(da'|a^*) \geq 0\}$$

# Entrant's Problem

- Size of potential entrants:  $M_t$
- One decision:
  - entry, after paying a sunk entry cost  $c^e$
- Entry decision:
  - enter if

$$\int v_t(a, \mu)g(da) \geq c^e$$

- free entry:

$$\int v_t(a, \mu)g(da) = c^e \text{ if } M_t > 0$$

# Distribution

Law of Motion:

$$\mu_{t+1}([0, a']) = \underbrace{\int_{a \geq a^*} F(a'|a) \mu_t(da)}_{\text{Continuing Incumbent}} + \underbrace{M_{t+1} G(a')}_{\text{Entrants}} \quad (1)$$

Define

$$\hat{P}_t = \begin{cases} \int_{a \in A} F(a'|a) & \text{if } a \geq a^* \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow$  Law of Motion:

$$\mu_{t+1} = \hat{P}_t \mu_t + M_{t+1} g \quad (2)$$



# Equilibrium

- Aggregate supply (endogenous)

$$Q^s(\mu_t) = \int q_t(a, \mu) \mu_t(da)$$

- Aggregate demand (exogenous)

$$Q^d$$

- Aggregate labor demand (endogenous)

$$N^d(\mu_t) = \int n_t(a, \mu) \mu_t(da)$$

- Aggregate labor supply (exogenous)

$$N^s$$

- Both markets clear at equilibrium
- Focus on stationary equilibrium
  - constant distribution over time

# Distribution

- Stationary Distribution:

$$\mu^* = \hat{P}\mu^* + M^*g \quad (3)$$

$$\Rightarrow \mu^* = M^*(I - \hat{P})^{-1}g \quad (4)$$

- stationary distribution is linearly homogeneous in  $m$  (scalar)
- stationary distribution can be found by simulation as well. (appendix)

# Counterfactual

- Entry cost parameter:  $c^e \uparrow$ 
  - expected discounted profits:  $\uparrow$
  - exit threshold  $a^*$ :  $\downarrow$
  - entrants mass  $m^*$ :  $\downarrow$
  - output price  $p^*$ :  $\uparrow$
  - entry rate/exit rate  $m^*/\mu^*$ :  $\downarrow$
  - firm-size distribution: ambiguous
    - price effect: incumbents increase output  $q^*$  and employment  $n^*$
    - selection effect: more incumbent firms are relatively-low productivity firms
    - selection effect: entrants are of better productivity

# Results

- Size effect
  - size of output  $\leftrightarrow$  size of employment  $\leftrightarrow$  productivity draw
  - unconditionally large firms have lower growth rate on average
- Age effect
  - unconditionally old firms have lower growth rate on average
  - firms age as they survive in the market over time
  - **no conditional age effect**
- Frictionless environment
  - model: young firms are small because they have lower draw on productivity
  - data: young firms are small not because they are inefficient
- Next Step: adding frictions (real, financial etc.)

# Cooley and Quadrini (2001)

- persistent shock + financial constraint  $\rightarrow$  size + age effect
  - conditional on age, the dynamics of firms are negatively related to the size of firms
  - conditional on size, the dynamics of firms are negatively related to the age of firms
- capture the features of the financial behavior of firm
  - small and younger firms pay fewer dividends, take on more debt, and invest more
  - small firms have higher values of Tobin's  $q$
  - investment of small firms is more sensitive to cash flows
- financial frictions
  - equity: cost or premium associated with increasing equity
  - debt: costly default
  - trade-off theory

## Intuition: firm's problem

- a stylized and simplified model
- decreasing return to scale production technology:

$$y = af(k + b)$$

- a: idiosyncratic productivity, i.i.d
  - k: owned capital (equity), no depreciation
  - b: borrowed capital (financed with debt)
- borrowing constraint:

$$b \leq k$$

- interest rate:  $r$
- value function:

$$v(k, b) = \max_{k', b'} af(k + b) - br - (k' - k) + \beta \int v(z', k') F(dz')$$

- efficient size:  $E\{af'(k^* + b^*)\} = r$

## Intuition: constrained firms

- optimal borrowing:

$$b' = k'$$

- capital accumulation:

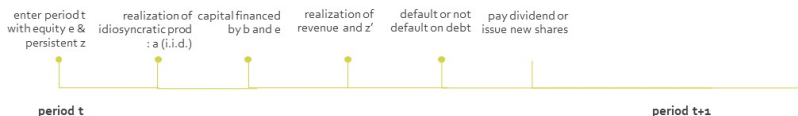
$$k' = af(2k) - rk + k$$

- growth rate:

$$\frac{k' - k}{k} = \frac{af(2k)}{k} - r$$

- decreasing in  $k$
- financial constraint impedes firms to jump directly to their efficient size.

# Full Model



- depreciation
- inter-temporal debt
- compound idiosyncratic shocks: persistent + transitory
- financial market frictions



# Firm's problem

- net worth end-of-period:

$$\pi(e, b, z + a) = (1 - \phi)(e + b) + (z + a)f(e + b) - (1 + \tilde{r})b$$

- a: transitory productivity (accidents), i.i.d, unexpected
- z: persistent productivity (technology), Markov process:  $z \rightarrow z'$ , revealed 1 period in advance
- e: equity (asset)
- $\phi$ : cost of capital (depreciation)
- $\tilde{r}$ : interest rate charged by intermediary
- endogenous default: threshold i.i.d shock  $\underline{a}$  implicitly defined by

$$\underbrace{(1 - \phi)(e + b) + (z + \underline{a})f(e + b) - (1 + \tilde{r})b}_{=\pi(e, b, z + \underline{a})} = \underline{e}(z')$$

- default if value of continuation is less than zero
- threshold net worth of default:  $\underline{e}(z')$
- $e(z') < \underline{e}(z') \Rightarrow$  liability renegotiated until  $e(z') = \underline{e}(z')$

## External Finance: Debt

- interest rate:

$$(1+r)b = (1+\tilde{r})b \int_{\underline{a}}^{\infty} g(da) + \int_{-\infty}^{\underline{a}} [(1-\phi)(e+b) + (z+a)f(e+b) - \xi]g(da)$$

- $r$ : risk-free interest rate
- $\xi$ : default loss
- $\Rightarrow$  threshold i.i.d. shock  $\underline{a} = \underline{a}(z, e, b, z')$ :

$$(1+r)b + \underline{e}(z') \int_{\underline{a}}^{\infty} g(da) + \xi \int_{-\infty}^{\underline{a}} g(da) = (1-\phi)(e+b) + h(\underline{a})F(e+b) \quad (5)$$

- where  $h(\underline{a}) = z + \underline{a} \int_{\underline{a}}^{\infty} g(da) + \int_{-\infty}^{\underline{a}} ag(da)$

## External Finance: Debt

- sequence of decisions: (a). default  $\rightarrow$  (b). equity issuance/ dividend payment  $\rightarrow$  (c). next period debt
- default does not lead to exit of the firm
- debt is re-negotiated after default
  - if  $\pi(e, b, z + a) < \underline{e}(z')$ , intermediary loss =  $\underline{e}(z') - \pi(e, b, z + a)$
- end-of-period net worth

$$q(e, b, z + a, z') = \begin{cases} \underline{e} + (a - \underline{a})f(e + b), & \text{if } a \geq \underline{a}(z, e, b, z') \\ \underline{e}, & \text{if } a \leq \underline{a}(z, e, b, z') \end{cases} \quad (6)$$

## External Finance: Equity

- sequence of decisions: (a). default  $\rightarrow$  (b). equity issuance/ dividend payment  $\rightarrow$  (c). next period debt
- equity finance:

$$d(x, e') = \begin{cases} x - e', & \text{if } x \geq e' \\ (x - e')(1 + \lambda), & \text{if } x \leq e' \end{cases} \quad (7)$$

- where  $x$ : end-of-period equity of the firm before (b)
- if  $d(x, e')$  is positive, firm pays dividend;
- if  $d(x, e')$  is negative, firm issues equity, at cost  $\lambda$ ;

## Firm's Problem

- sequence of decisions: (a). default  $\rightarrow$  (b). equity issuance/ dividend payment  $\rightarrow$  (c). next period debt
- value of the firm at the end of the period after (b) but before (c):

$$\Omega(z, e) = \max_b \left\{ \beta \sum_{z'} \int_{\underline{a}} \tilde{\Omega}(z', q((e, b, z + a, z'))) \Gamma(z'|z) f(da) \right\} \quad (8)$$

s.t. equation (5) and (6)

- where  $\tilde{\Omega}(z, e)$ : end-of-period value after (a) but before (b) s.t.

$$\tilde{\Omega}(z', \underline{e}) = 0$$

s.t.

$$\tilde{\Omega}(z', x) = \max_{e'} \{d(x, e') + \Omega(z', e')\}$$

s.t. equation (7)

## Proposition 3

PROPOSITION 3: There exists a unique function  $\Omega^*(z, e)$  that satisfies the functional equation (8). In addition, if for  $a_1$  and  $a_2$  sufficiently small,  $g(a) < a_1$  for all  $a < -a_2$ , then

- the firm's solution is unique, and the policy rule  $b(z, e)$  is continuous in  $e$ ;
- the input of capital  $k = e + b(z, e)$  is increasing in  $e$ ;
- there exist functions  $\underline{e}(z) < \hat{e}(z) < \bar{e}(z)$ ,  $z \in Z$ , for which the firm renegotiates the loan if the end-of-period resources are smaller than  $\underline{e}(z)$ , will issue new shares if they are smaller than  $\hat{e}(z)$ , and distribute dividends if they are bigger than  $\bar{e}(z)$ ;
- the value function  $\Omega^*(z, e)$ , is strictly increasing and strictly concave in  $[e, \bar{e}]$ .

## Proposition 3: Comment

There exist functions  $\underline{e}(z) < \hat{e}(z) < \bar{e}(z)$ :

- if  $e < \hat{e}(z)$ : the firm issues new shares to increase equity level to  $\hat{e}(z)$ , as marginal increase in value w.r.t.  $e > 1 + \lambda$
- if  $\hat{e}(z) < e < \bar{e}(z)$ : the firm will not issue new shares, as marginal increase in value w.r.t.  $e < 1 + \lambda$
- if  $\bar{e}(z) < e$ : the firm distribute dividends, as marginal increase in value w.r.t.  $e < 1$
- who issues equity?
  - with relatively lower net worth
  - with improvement in technology

# Entrants

- new firms are created with an initial value of equity raised by issuing new shares to an optimal size:  $\hat{e}(z)$
- the cost of creating a new firm with initial productivity  $z$ :

$$\kappa + (1 + \lambda)\hat{e}(z)$$

- surplus of entry:

$$\Omega(z, \hat{e}(z)) - \kappa - (1 + \lambda)\hat{e}(z)$$

- free entry (general equilibrium property)

$$\Omega(z_N, \hat{e}(z_N)) = \kappa + (1 + \lambda)\hat{e}(z_N)$$

- invariant measure of firms  $\mu^*$  exists.



## i.i.d shock: role of financial friction

- $z$  takes only two values: an absorbing shock  $z_0 = 0$  and  $z_1$
- conditional on surviving, the shock is i.i.d.
- isolate the financial mechanisms from persistence mechanism
- key properties of the financial behavior of firm
  - small firms take on more debt (higher leverage).
  - small firms face higher probability of default.
  - small firms have higher rates of profits.
  - small firms issue more shares and pay fewer dividends.
- key properties of firm dynamics
  - small firms grow faster and experience higher volatility of growth.
  - small firms face higher probability of default.
  - small firms experience higher rates of job reallocation.
  - *without conditioning on size*, young firms experience higher rates of growth, default, and job reallocation.

## persistent shock: interaction with financial friction

- conditional on surviving,  $z$  follows a symmetric two-state Markov process
- firms differ over two dimensions: equity and productivity
- conditional on equity size, high productivity firms borrow more and implement larger production scales. trade-off:
  - a larger production scale allows higher expected profits
  - a larger production scale implies higher volatility of profits
- size dependence and age dependence in the dynamics of firm
  - unconditionally and conditionally
  - size dependence as before
  - age dependence derives from heterogeneous composition of firm types in each age class of firms
  - effect of entry: initial productivity of new firms
  - effect of persistent shock
  - age effect is more important for small firms; and it almost disappears for very large firms.

# Conclusion

- Hopenhayn (1992)
  - frictionless environment (except fixed cost)
  - size dependence
  - no conditional age dependence
- \*Jovanovic (1982)
  - learning model
  - age dependence
  - no conditional size dependence
- Cooley and Quadrini (2001)
  - financial friction and persistent effect
  - conditional size dependence
  - conditional age dependence
  - reconcile size-dependent financial features
- **Next Step:**
  - richer heterogeneity
  - business cycle